LEONHARDI EULERI
OPERA OMNIA

LEONHARDI EULE

OPERA OMNIA

SUB AUSPICIIS
SOCIETATIS SCIENTIARUM NATURA
HELVETICAE

EDENDA CURAVERUNT

ANDREAS SPEISER
LOUIS GUSTAVE DU PASQUIER
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SERIES PRIMA

OPERA MATHEMATICA

VOLUMEN VICESIMUM SECUNDUM

AUCTORITATE ET IMPENSIS
SOCIETATIS SCIENTIARUM NATURALIUM HEL

BASILEAE MCMXXXVI

VENDITIONI EXPONUNT
ORELL FÜSSLI TURICI ET LIPSIAE
B. G. TEUBNER LIPSIAE ET BEROLINI

MMENTATIONES ANALYTICAE

AD THEORIAM AEQUATIONUM DIFFERENTIALIUM PERTINENTES

EDIDIT

HENRI DULAC

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IETATIS SCIENTIARUM NATURALIUM HELVETICAE

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VENDITIONI EXPONUNT
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TYPIS EXCUSSIT ORELL FÜSSLI TURICI

PRÉFACE DE L'ÉDITEUR

Les volumes 22 et 23 de la première série de la collection LEONHARDI EULERI *O*
nia rassemblent les divers mémoires d'EULER traitant plus particulièrement
oblèmes relatifs aux équations différentielles et aux équations aux dérivées parti
e partie des questions traitées dans ces mémoires sont exposées, sous une forme
néral différente, dans les trois volumes des *Institutiones calculi integralis*. D'au
estions importantes relatives aux mêmes sujets ne sont exposées que dans le *Calc*
egralis.

En 1726, lorsque paraissent les premiers travaux d'EULER, les cas d'intégrab
l'équation de RICCATI venaient d'être publiés, la méthode de la séparation des varia
tégration de l'équation homogène du premier ordre, de l'équation linéaire, de l'équa
BERNOULLI, l'emploi, dans certains cas particuliers, d'un facteur intégrant ou m
cateur, étaient connus, ainsi que, pour les équations différentielles d'ordre supér
cas de réduction au premier ordre et l'intégration de certaines équations linéai
mpossibilité d'exprimer par des fonctions usuelles toutes les quadratures, a
ntré qu'on ne pouvait, que dans des cas très particuliers, obtenir l'intégration expl
équations différentielles au moyen de fonctions connues. Les résultats acquis
ttaient encore d'espérer que l'on pourrait obtenir cette intégration par des quadrat

Les travaux d'EULER ont apporté une contribution importante à l'intégration
nations différentielles, mais ils paraissent avoir surtout une importance historiqu
orique. Partant, en effet de solutions ou de procédés employés dans des cas
rticuliers, EULER en a dégagé des méthodes générales d'intégration. Il est évident
n'est qu'en raison de la stérilité relative de ces méthodes, que, bien avant de pou
démontrer, on a admis l'impossibilité d'intégrer les équations différentielles pa
adratures. Le nombre restreint de cas d'intégrabilité nouveaux, obtenus par l'E

1) Voir, par exemple, pour ces questions: *Encyclopédie des Sciences mathématiques*, T. II. vol. 3 p. 64. Paris et Leipzig 1910.

le problème d'intégration à des problèmes plus simples de même ...
fournissant d'intégration effective que dans des cas encore plus particul...

Dans le mémoire 10 (d'après les numéros d'Eneström) EULER indic... cas de réduction d'équations du deuxième ordre au premier ordre. Ce... classiques.

Une série de mémoires sont consacrés à la méthode appelée par EU... per quadratura curvarum"[1]. Conduit fortuitement, ainsi qu'il l'indique dans... à la représentation d'une solution $y(x)$ d'une équation différentielle ... intégrale définie dans laquelle x figure comme paramètre, EULER a cherc... ploi systématique de ce mode de représentation, dont il paraît avoir l... l'exemple, et dont les applications bien connues ont été faites, en particul... GAUSS, KUMMER. EULER emploie cette méthode de deux manières ... le mémoire 31[2]), et dans le chapitre XI de la 1re partie du 2e volume du C... il obtient d'abord la solution considérée sous forme de série et évalue ... de cette série au moyen d'une intégrale de l'espèce indiquée. EULER a... une méthode plus directe, en formant l'équation différentielle vérifiée p... définie donnée, dans laquelle x figure comme paramètre. Cette méthode ... les mémoires 44 et 45, appliquée ensuite dans 70, 274 ainsi que dans ... la 1re partie du 2e volume du *Calculus integralis*.

On peut rattacher au même ordre d'idées (détermination d'une fo... d'opérations données effectuées sur une courbe) certains des résultats ... où EULER donne des méthodes graphiques pour l'intégration de certaine... particulier de l'équation de RICCATI.

EULER a donné un développement important au procédé du multiplica... une véritable méthode d'intégration. Les mémoires 260, 430 sont cons... de cette méthode pour l'intégration des équations du premier ordre. Le... 429, 431, 700 traitent de son emploi pour les équations du deuxième o... qui est, en grande partie reproduit dans les chapitres II et III de la 2me s... lume du *Calculus integralis*, nous trouvons un exposé complet de l'intégrat... usuelles du premier ordre, la plupart des résultats énoncés et des exem... restés dans l'enseignement.

EULER emploie également de deux façons différentes la méthode d... On bien, partant d'une équation différentielle donnée, il cherche à la ...

---

1) Voir la note de la page 16.
2) Le mémoire 11, relatif à la même question, ne fait qu'énoncer les résulta...

si découverts sont relatifs à des équations de formes assez particulières, mais l'im...ce de cette notion de multiplicateur a été nettement montrée par EULER. Il a... r, en effet, comment par son emploi, on retrouve tous les cas d'intégrabilité con... mment la connaissance d'un multiplicateur permet d'abaisser d'une unité l'ordre d'... uation et comment la connaissance de deux multiplicateurs pour une équation ... cond ordre permet de ramener son intégration à des quadratures.

Les mémoires 595 et 751, montrent par deux méthodes différentes, comment l'em... fractions continues permet d'obtenir, pour n quelconque, l'intégration de l'équa... RICCATI

$$\frac{dy}{dx} + ay^2 = bx^n \qquad\qquad a \text{ et } b \text{ constants}$$

d'en déduire tous les cas où l'intégrale s'exprime en termes finis.

Dans les mémoires de ces volumes 22 et 23, EULER emploie fréquemment des sé... is je n'ai pu relever de cas où une série soit donnée comme l'expression définitive d'... ution d'une équation différentielle. Ou bien, comme nous l'avons déjà indiqué, la s... un intermédiaire conduisant à une autre expression de la solution considérée, ou b... mme dans 284, EULER indique explicitement qu'il n'utilise les développements ... ns les cas d'intégrabilité où le nombre de leurs termes est fini.

Ce n'est pas là, du reste un principe constant chez EULER, car il s'en écarte dan... pitres VII et VIII de la première partie du deuxième volume du *Calculus integ...* blié postérieurement à 284.

L'application des méthodes précédentes a conduit EULER à divers cas d'intégrab... uveaux, aussi bien qu'à d'élégantes démonstrations de cas d'intégrabilité déjà con... ons les diverses espèces d'équations qu'il a le plus fréquemment considérées.

L'équation (1) dans les mémoires 11, 31, 51, 70, 95, 269, 284, 595, 751.

Des équations de RICCATI de la forme générale

$$\frac{dy}{dx} + P(x)y^2 + Q(x)y + R(x) = 0$$

ns les mémoires 51, 70, 95, 265, 269, 678, 734. L'équation

$$y\frac{dy}{dx} + P(x)y + Q(x) = 0$$

ns les mémoires 269 et 430.

$$(a + bx^2)x^2d^2y + (c + fx^2)xdxdy + (g + hx^2)ydx$$

$$(ax^2 + bx + c)d^2y + (fx + g)dxdy + hydx^2 = 0$$

dans les numéros 95, 274, 284, 431, 677, 678, en vue le plus souvent d'en
à l'intégration des équations de RICCATI.

Nous avons laissé de côté dans ce qui précède les mémoires rela
des équations linéaires d'ordre quelconque. Dans le mémoire 720 EU
l'on a intégré l'adjointe de LAGRANGE d'une équation différentielle
quelconque $P(y) = 0$, la solution de l'équation linéaire non homogène
tient par des quadratures.

Les mémoires 62 et 188 exposent les méthodes d'intégration des
à coefficients constants: le premier pour les équations homogènes,
équations avec second membre. Ce dernier cas est encore traité dans
mémoire 680, où EULER étudie les équations de LAGRANGE et certai
de ces équations. Antérieurement, dans 236, l'étude des équations d
solutions singulières avait été abordée sur des exemples d'un caractè

Les formules rencontrées dans l'intégration des équations linéai
ont conduit EULER à étudier dans 679 les transformations des expr

$$\int_0^x pdx \int_0^x qdx \int_0^x rdx \ldots \int_0^x sdx \int_0^x tdx$$

renfermant un nombre quelconque de signes d'intégrations superposé
étant des fonctions données de x.

En particulier pour $p = q = r = \ldots = s = t$ l'expression est é
des termes contenant chacun un seul signe d'intégration. La formu
été employée dans le mémoire 681 dont le titre indique l'intégrati
différentielle d'ordre fractionnaire. EULER n'a pu, faute de notations
dans toute sa généralité la formule qu'il obtient, qui n'est autre q
connu:

$$y = \int_0^x (x - z)^{q-1} X(z)\, dz$$

représente, si q est un entier, une solution de l'équation

$$\frac{d^q y}{dx^q} = q!\, X(x)$$

ressantes, en employant dans le cas où q est fractionnaire des formules démontr-
r q entier.

Des problèmes relatifs à la rectification des courbes et en particulier le probl-
ité par J. BERNOULLI et HERMANN de la recherche de courbes algébriques rectifia-
conduit EULER à étudier dans les mémoires 48, 245, 622, 650, 779 des questions d'a-
indéterminée. Les questions traitées dans ces mémoires rentrent dans le probl-
néral suivant: Etant données un certain nombre de fonctions $P(x,y)$, $Q(x,y)$, ..., $S(x$
blir entre x et y une relation telle que les intégrales $\int P(x,y)dx$, $\int Q(x,y)dx$,
$(x,y)dx$ s'expriment simultanément au moyen de quadratures données ou, comme
rticulier, soient intégrables.

EULER applique notamment ses méthodes à la recherche de courbes rectifiables
nt les arcs satisfont à certaines conditions.

On peut rattacher en partie à l'analyse indéterminée et en partie aux applicat-
la théorie du multiplicateur le numéro 856, où il s'agit de trouver une courbe ta-
rone pour un mouvement dans un milieu résistant. Le problème traité dans 784, con-
blème d'analyse indéterminée, peut être ramené à l'intégration d'une équation liné-
x dérivées partielles du premier ordre.

Le mémoire 322 est en majeure partie consacré à des considérations sur les princ-
l'Analyse et l'emploi des fonctions discontinues, mais une intégration d'équation
ivées partielles traitée dans ses dernières pages, le rattache aux mémoires consa-
EULER à ces équations et dont il nous reste à parler. Dans 285, de nombreux
ntégration d'équations aux dérivées partielles du premier ordre sont traités.

EULER n'établit pas de méthode générale d'intégration, il se sert de l'intégra-
parties, et de la remarque suivante: $V(x,y)dU$ n'est intégrable que si V est fonc-
U. On ne peut qu'admirer avec quelle habileté, par des artifices assez divers, il réu-
ntégrer la plupart des équations que nous savons intégrer. Les calculs auxque-
outit, sont en général ceux qui résultent de la recherche d'une intégrale complète
procédés classiques. Les mémoires que nous n'avons pas encore cités traitent de l'i-
tion de certaines classes d'équations aux dérivées partielles du second ordre ou d'o-
périeur. Etudiant dans 319 l'équation

$$\frac{\partial^2 z}{\partial t^2} = a^2 \frac{\partial^2 z}{dx^2} + \frac{b}{x}\frac{\partial z}{\partial x} + \frac{c}{x^2} z$$

LER en montre les analogies avec l'équation de RICCATI, dans la recherche des
ntégrabilité. Le mémoire 737 contient une théorie générale de l'emploi des changem-

problème des cordes vibrantes traité également dans 318.

Les mémoires 724 et 785 donnent l'intégration complète de certaine

tions linéaires aux dérivées partielles d'ordre quelconque, mais de formes t

Enfin, dans 741, EULER a cherché à étendre aux équations linéair

partielles à coefficients constants sa méthode d'intégration des équatio

linéaires à coefficients constants. Il obtient ainsi, dans certains cas, l'in

de ces équations. Les raisonnements employés manquent parfois de rigu

Il serait injuste de reprocher à EULER d'être resté fidèle aux habitu

dans certains raisonnements et de ne pas avoir toujours donné à ceux-ci l

aujourd'hui. Ces habitudes étaient tout à fait dans la nature des choses

pour le développement de l'Analyse.

On comprendrait mal que, placés devant l'immense domaine que le

méthodes nouvelles, les mathématiciens du 18^{me} siècle au lieu d'exp

comme ils l'ont fait, ces régions inconnues se fussent tout d'abord occup

théories préliminaires pour leurs études. L'exploration du champ nouvea

lyse permettait seule de déterminer quelles seraient les théories utiles et h

il conviendrait de les développer. Au reste, ce n'est guère que dans les q

viennent des séries que l'on trouve chez EULER des raisonnements dou

du prolongement analytique implicitement admise par EULER fournit c

la justification des inductions hardies que l'on rencontre dans certains

Si quelques raisonnements d'EULER paraissent incomplets, cela s

que certaines façons de raisonner, peu usitées aujourd'hui, étaient

18^{me} siècle, que les auteurs avaient lieu de croire que les raisonnements s

facilement rétablis par les lecteurs. Je n'ai pu relever aucun cas où une affi

soit en défaut, lorsqu'il indique dans le cours d'un raisonnement qu'i

necessaire sans le démontrer. Souvent, bien que, ni ses raisonnements, n

n'indiquent la solution trouvée d'un problème comme la solution la

peut constater qu'il a bien obtenu cette solution générale. EULER a

certaines assertions fondées sur des raisonnements qu'il avait donné

même temps leur manque de rigueur.

Le rapide exposé qui précède permet à peine de voir quelle est l

blèmes abordés par EULER dans ce seul domaine des équations diff

équations aux dérivées partielles. Dans la plupart des cas, ou bien EUL

à aborder les problèmes qu'il traite, ou bien il en a donné des solution

mémoires de ces deux volumes suffisent à eux seuls pour donner un

progrès qui sont dus à EULER, soit dans les notations, soit dans les mé

nsi se rendre compte de la grande importance de ses travaux dans l'élaboration es théories relatives aux équations différentielles et aux équations aux dérivées

, le 16 juin 1924.

H. Dulac.

INDEX

Insunt in hoc volumine indicis ENESTROEMIANI commentationes
10, 11, 31, 44, 45, 48, 51, 62, 70, 95, 188, 236, 245, 265, 269, 274,

70. De constructione aequationum . . . .

Commentarii academiae scientiarum Petropolitanae 9 (1737), 1744, p. 85—97

95. De aequationibus differentialibus, quae certis tantum casibus inte-
grationem admittunt . . . .

Commentarii academiae scientiarum Petropolitanae 10 (1738), 1747, p. 40—55

188. Methodus aequationes differentiales altiorum graduum integranc
ulterius promota . . . .

Novi Commentarii academiae scientiarum Petropolitanae 3 (1750/1), 1753, p. 3—3

236. Exposition de quelques paradoxes dans le calcul intégral . . .

Mémoires de l'académie des sciences de Berlin 12 (1756), 1758, p. 300—321

245. De methodo DIOPHANTEAE analoga in analysi infinitorum . . .

Novi Commentarii academiae scientiarum Petropolitanae 5 (1754/5), 1760, p. 84—14

265. De aequationibus differentialibus secundi gradus . . . .

Novi Commentarii academiae scientiarum Petropolitanae 7 (1758/9), 1761, p. 163—20

269. De integratione aequationum differentialium . . . .

Novi Commentarii academiae scientiarum Petropolitanae 8 (1760/1), 1763, p. 3—6

274. Constructio aequationis differentio-differentialis
$Aydu^2 + (B + Cu)dudy + (D + Eu + Fuu)\,ddy = 0$, sumto ele-
mento du constante . . . .

Novi Commentarii academiae scientiarum Petropolitanae 8 (1760/1), 1763, p. 150—15

284. De resolutione aequationis $dy + ayydx = bx^m dx$ . . . .

Novi Commentarii academiae scientiarum Petropolitanae 9 (1762/3), 1764, p. 154—16

NOVA METHODUS INNUMERABILES AEQUATIONES DIFFERENTIALES SECUND GRADUS REDUCENDI AD AEQUATIONES DIFFERENTIALES PRIMI GRADUS

Commentatio 10 indicis ENESTROEMIANI
Commentarii academiae scientiarum Petropolitanae 3 (1728), 1732, p. 124—137

1. Quando ad aequationes differentiales secundi vel altioris cui
lus perveniunt analytae, in iis resolvendis duplici modo versantur.
uirunt, an in promtu sit eas integrare; id si fuerit, obtinuerunt,
derabant. Cum autem integratio vel prorsus impossibilis, vel saltem
or videtur, conantur eas ad differentiales primi gradus reducere; c
quibus facilius iudicari potest, an construi queant, nullaeque aequa
erentiales, nisi primi gradus, adhuc cognitis methodis construi p
od ad illud attinet, de eo hac dissertatione explicare non est propos
modo autem aequationes differentiales altiorum graduum praesertim
ndi ad differentiales primi gradus sint reducendae, methodum qua
ue inusitatam, et quae latissime patet, in sequentibus sum expositu

2. Iam quidem saepenumero Mathematici, quando aequationes
tiales secundi vel altiorum graduum occurrerunt, eas ad differen
ni gradus reduxerunt, atque deinde construxerunt; quemadmodum
; in constructionibus catenariae, elasticae, proiectoriae in medio
que resistenti pluriumque aliarum curvarum, quarum aequationes
erentiales secundi vel tertii gradus sunt inventae. Pleraeque quidem

differentiales primi gradus fuerant reductae. Earum aut
ratio ita est comparata, ut vel utraque vel saltem alterut
ipsa desit, earum eiusve differentialibus et differentio-differ
tiones tantum ingredientibus.

3. Si autem in aequatione differentio-differentiali alterut
caret, facile est eam ad simpliciter differentialem reducere
differentialis quantitatis deficientis factum ex nova quadam
alterum differentiale. Hac enim ratione, si constans quod
fuerit positum, differentio-differentiali aequale invenitur si
tiale; quo substituto aequatio habetur differentialis primi g
aequatione

$$Pdy^n = Qdv^n + dv^{n-2}ddv,$$

ubi P et Q significant functiones quascunque ipsius y, a
ponitur. Quia ipsa v non ingreditur aequationem, fiat dv
$ddv = dzdy$. His substitutis ista oritur aequatio

$$P\,dy^n = Qz^n dy^n + z^{n-2}\,dy^{n-1}\,dz,$$

divisaque hac per dy^{n-1} ista

$$P\,dy = Qz^n\,dy + z^{n-2}\,dz,$$

quae est simpliciter differentialis.

4. Alias aequationes differentio-differentiales, nisi huius
quantum scio, ad differentiales primi gradus unquam red
promtu fuerit eas prorsus integrare. Hic autem methodu
non quidem omnes, sed tamen innumerabiles aequatione
rentiales utut ab utraque indeterminata affectae ad simpli
reduci poterunt. Ita vero in iis reducendis versor, ut eas ce
tutione in alias transformem, in quibus alterutra indeter
facto ope substitutionis paragrapho praecedente exposita
penitus ad differentiales primi gradus reducentur.

5. Cum observassem eam esse quantitatum exponent
earum dignitatum, quarum exponens est variabilis mane
vata constante, proprietatem, ut si differentientur, denuo

ale $c^x dx$, differentio-differentiale $c^x(ddx + dx^2)$, ubi x nonnisi in exponent
greditur. Haec considerans perspexi, si in aequatione differentio-different
eo indeterminatarum huiusmodi exponentialia substituantur, tum ip
riabiles tantummodo in exponentibus superfuturas esse. Quo cognito op
t, ut ea exponentialia loco indeterminatarum substituenda ita accomm
ntur, ut facta substitutione ea divisione tolli queant; hoc modo alter
ltera indeterminata ex aequatione eliminabitur, eiusque duntaxat differ
alia supererunt.

6. Haec quidem operatio non in omnibus aequationibus succedit; veru
men eam tria aequationum differentialium 2^{di} gradus genera admittere
rvavi. Primum genus est omnium earum aequationum, quae nonnisi duo
nstant terminis. Alterutrum eas comprehendit aequationes, in quarum s
lis terminis indeterminatae aequalem dimensionum numerum constitu
que vero indeterminata ipsa solum, sed etiam eius differentialia cuius
adus dimensionem unam constituere existimanda sunt. Ad tertium ge
s refero aequationes, in quarum singulis terminis alterutra indetermin
ndem obtinet dimensionum numerum; quorsum eadem pertinent, q
odo de aestimatione dimensionum allata sunt[2]). Omnes igitur aequatio
l haec tria genera pertinentes hic reducere docebo.

7. Omnes aequationes ad primum genus pertinentes sub hac gene
rmula comprehenduntur:

$$ax^m dx^p = y^n dy^{p-2} ddy,$$

bi dx constans ponitur. Et si enim in aequatione quapiam neque dx neque
nstans accipiatur, sed aliud quoddam differentiale inde pendens, id n
fficultatis habet, cum cognita sit methodus, quod constans erat differenti
riabile faciendi et vice eius aliud quoddam constans. Ad hanc vero aeq
onem reducendam pono

$$x = c^{av} \text{ et } y = c^v t.$$

1) In primis suis operibus, usque ad annum 1734, utitur EULERUS littera c loco e. H.

2) Vide *Institutionum calculi integralis* vol. II, § 819, 790—811, 822—830; LEONHARDI EU
era omnia, series I, vol. 12. H.

Atque hinc

$$ddx = ac^{av}(ddv + adv^2)$$

et

$$ddy = c^v(ddt + 2\,dtdv + tddv + tdv^2).$$

Sed cum dx ponatur constans, erit $ddx = 0$ adeoque ddv …
stituto loco ddv habebitur

$$ddy = c^v(ddt + 2\,dtdv + (1-a)tdv^2).$$

Surrogentur hi valores loco x et y in aequatione proposita, t…
ca in hanc

$$ac^{av(m+p)}a^p dv^p = c^{(n+p-1)v}t^n(dt + tdv)^{p-2}(ddt + 2dtdv + \ldots$$

8. Iam a determinari debet ita, ut exponentialia division…
Hoc ut fiat, oportet sit

$$av(m+p) = (n+p-1)v,$$

inde colligitur $a = \frac{n+p-1}{m+p}$. Superior igitur aequatio deter…
in sequentem

$$a\left(\frac{n+p-1}{m+p}\right)^p dv^p = t^n(dt + tdv)^{p-2}(ddt + 2\,dtdv + \frac{m \ldots}{m \ldots}$$

Quae protinus ex proposita orta fuisset, si posuissem

$$x = c^{(n+p-1)v:(m+p)} \text{ et } y = c^v t.$$

Est autem $n+p-1$ numerus dimensionum, quas y constituit, et…
Facile ergo in quovis casu particulari a determinatur statimque…
tutio habebitur. In aequatione inventa, cum absit v, ponatur…

$$ddv = zddt + dzdt,$$

sed

$$ddv = -adv^2 = \frac{1-n-p}{m+p}z^2dt^2.$$

Hinc invenitur

$$ddt = \frac{-dzdt}{z} + \frac{1-n-p}{m+p}zdt^2.$$

$$= t^n (dt + tzdt)^{p-2}\left(\frac{1-n-p}{m+p} zdt^2 - \frac{dzdt}{z} + 2zdt^2 + \frac{m-n+1}{m+p}\, tzzdt^2\right).$$

divisa per dt^{p-1} dabit

$$\left(\frac{n+p-1}{m+p}\right)^p z^p dt = t^n (1+tz)^{p-2}\left(\frac{1+2m-n+p}{m+p} zdt - \frac{dz}{z} + \frac{m-n+1}{m+p} tz^2 dt\right)$$

). Reducta ergo est aequatio generalis proposita

$$ax^m dx^p = y^n dy^{p-2} ddy$$

anc differentialem primi gradus

$$\left(\frac{n+p-1}{m+p}\right)^p z^{p+1} dt = t^n (1+tz)^{p-2}\left(\frac{1+2m-n+p}{m+p} z^2 dt + \frac{m-n+1}{m+p} tz^3 dt - dz\right)$$

iplicata aequatione inventa per z. Haec aequatio unico actu ex ea inveni t, posito in prima substitutione loco v hoc $\int z dt$. Fieri ergo debet

$$x = c^{(n+p-1)\int z dt : (m+p)}$$

co y poni debet $c^{\int z dt} t$ sive, quod eodem redit, ponatur

$$x = c^{(n+p-1)\int z dt} \text{ et } y = c^{(m+p)\int z dt} t.^{1)}$$

aequatione differentiali inventa iterum proposita differentialis secund us inveniri debeat, videamus, quales loco z et t substitutiones adhibe ant. Cum sit $x = c^{(n+p-1)\int z dt}$, erit $c^{\int z dt} = x^{1:(n+p-1)}$, quare $y = x^{(m+p):(n+p-1)}$ habetur $t = yx^{-(m+p):(n+p-1)}$. Deinde quia $c^{\int z dt} = x^{1:(n+p-1)}$

$$\int z dt = \frac{1}{n+p-1}\, lx;$$

$$z dt = \frac{dx}{(n+p-1)x}.$$

---

1) In his formulis z denotat numerum praecedentem z multiplicatum per $m+p$. H. D.

Sed est[1])

$$dt = x^{(m+1)(n+p-1)} dy \quad \frac{m}{n} \quad \frac{p}{p-1} y \ldots$$ [illegible]

Consequenter invenietur

$$z = dx : [(n+p-1) x^{(m+n+1)(n+p-1)} dy \quad (m \quad p \ldots$$ [illegible]

Perspicuum autem est, si z in t vel t in [illegible] detur, etiam rel[illegible] inter se habeant, inveniri posse.

10. Illustremus haec, quae generaliter inventa sunt [illegible] particulari. Sit

$$xdxdy = yddy,$$

quae reducitur dividendo per dy ad hanc

$$xdx = ydy^{-1}ddy.$$

Huic generali accommodata, habebitur $a = 1$, $m = 1$, p [illegible] tutis his in aequatione differentiali primi gradus (§ 9), h[illegible] proposita reducitur,

$$\tfrac{1}{2}z^2dt = t(1+tz)^{-1}(\ldots) dt \quad \tfrac{1}{2}t\ldots dt$$ [illegible]

quae abit in

$$z^2dt + tz^2dt \quad 3t\ldots dt \quad t^2\ldots dt \quad 2t\ldots$$ [illegible]

Ad hanc aequationem proposita $xdxdy = yddy$ reduc[illegible]

$$x = e^{\int z dt + 2} \text{ et } y = e^{\int z dt} t.$$ [illegible]

Constructio ergo aequationis propositae pendet a const[illegible] differentialis inventae; haec si construi poterit, et ea [illegible] reipsa integrabilis, ea quoque integrari poterit.

11. Secundum genus aequationum differentio-differen[illegible] methodo ad differentiales primi gradus reducere possum[illegible] quae in singulis terminis eundem dimensionum, quas inde[illegible] differentialia constituunt, numerum tenent. Aequatio ge[illegible] est sequens

$$ax^m y^{-m-1} dx^p dy^{2-p} + bx^n y^{-n-1} dx^q dy^{2-q} \ldots$$ [illegible]

1) Editio princeps hoc $m+n+2p-1$ habet $m \quad n \quad 2p$ [illegible]

2) Editio princeps: $x = e^{\int zdt}$ et $y = e^{2\int zdt} t$. Si haec mutatio [illegible] in aequatione differentiali z per $2z$ mutare.

...en quodcumque noluerit insuper addi possunt, operatio enim eadem man...
sent adhuc addi $ex^r y^{-r-1} dx^q dy^{2-q}$ et huiusmodi quotquot libuerit; pro
mpla particularia, ad quae reducenda generalis accommodari debet, plurib
cioribusve constant terminis. Tres vero terminos, ut dixi, assumsis
icit, cum plures alium reducendi modum non requirant.

12. Aequationem propositam reduco substituendis loco x, c^v et loco y, c
n igitur sit

$$x = c^v \text{ et } y = c^v t,$$

$$dx = c^v dv \text{ et } dy = c^v(dt + tdv)$$

roque

$$ddx = c^v(ddv + dv^2)$$

$$ddy = c^v(ddt + 2dtdv + tdv^2 + tddv).$$

a vero dx ponitur constans, erit $ddx = 0$, hinc igitur $ddv = -dv^2$, ha
rem habebitur

$$ddy = c^v(ddt + 2dtdv).$$

nantur hi valores in aequatione loco x, y, dx, dy et ddy, transformabit
in sequentem:

$${}^v t^{-m-1} dv^p (dt + tdv)^{2-p} + bc^v t^{-n-1} dv^q (dt + tdv)^{2-q} = c^v(ddt + 2dtdv)$$

ae divisa per c^v abibit in hanc

$$at^{-m-1} dv^p (dt + tdv)^{2-p} + bt^{-n-1} dv^q (dt + tdv)^{2-q} = ddt + 2dtdv.$$

hac cum desit v, pono $dv = zdt$, erit

$$ddv = zddt + dzdt,$$

$ddv = -dv^2 = -z^2 dt^2$, ergo

$$ddt = -zdt^2 - \frac{dzdt}{z}.$$

Hinc ista obtinebitur aequatio:

$$at^{-m-1}z^p dt^p (dt + ztdt)^{2-p} + bt^{-n-1}z^q dt^q (dt + ztdt)^{2-q} = zd$$

seu haec ordinatior

$$at^{-m-1}z^p dt(1 + zt)^{2-p} + bt^{-n-1}z^q dt(1 + zt)^{2-q} = zd$$

13. Aequatio haec differentialis primi gradus unico ne
elici potuisset, si statim positum esset

$$x = c^{\int zdt} \text{ et } y = c^{\int zdt} t;$$

unde foret

$$dx = c^{\int zdt} zdt \text{ et } dy = c^{\int zdt}(dt + tzdt)$$

atque

$$ddx = c^{\int zdt}(zddt + dzdt + zzdt^2) = 0,$$

quare $ddt = -zdt^2 - dzdt:z$. Hoc in usum vocato habebitu

$$ddy = c^{\int zdt}(zdt^2 - dzdt:z).$$

Propositum sit hoc exemplum

$$y^{a+1}ddy = x^a dx^2,$$

mutetur id in

$$ddy = x^a y^{-a-1} dx^2.$$

Collato hoc cum generali aequatione fiet $a = 1$, $b = 0$, $m = a$,
hoc exemplum, ut generalis formula, reducatur, haec invenietu

$$t^{-a-1}z^2dt = zdt - dz : z.$$

Sive haec

$$t^{-a-1}z^3dt = z^2dt - dz.$$

Quae si constructionem admitteret, et differentialis secundi
construi posset. Notandum est semper fere ad eiusmodi aequat
tiales perveniri, quae admodum difficulter vel prorsus non const

14. Assumo aliud exemplum,

$$xdxdy - ydx^2 = y^2ddy,$$

quod ad modum generalis aequationis hanc induit formam

$$xy^{-2}dxdy - y^{-1}dx^2 = ddy.$$

ondet ergo exemplo proposito sequens aequatio differentialis

$$t^{-2}zdt(1+zt) - t^{-1}z^2dt = zdt - dz:z.$$

plicetur haec per t^2z, habebitur

$$z^2dt + z^3tdt - z^3tdt = z^2t^2dt - t^2dz$$

$$z^2dt = z^2t^2dt - t^2dz,$$

separata dat

$$dz:z^2 = dt(t^2-1):tt$$

tegrata hanc

$$-1:z = t + 1:t - a \quad \text{sive} \quad atz - t = t^2z + z.$$

vero $z = dv:dt$. Itaque

$$atdv - tdt = t^2dv + dv$$

$v = tdt:(at - tt - 1)$. Quia vero $c^v = x$, erit $v = lx$ et $t = y:x$, ergo

$$dv = dx:x \text{ et } dt = (xdy - ydx):xx,$$

equenter

$$ydy + xdx = aydx.$$

aequatio iterum integrari potest, cum vero tantum noto casum, quod = 0 ea transeat in aequationem circuli.

15. Accipio nunc casum, quo plures, quam in generali aequatione, sin ini

$$dx^3 + xxdy^3 - yxdxdy^2 - yxdx^2dy + yx^2dxddy - y^2xdxddy = 0.$$

exemplum modo supra exposito reducere licebit. Cum dx ponatur con , maneant eaedem substitutiones scilicet

$$x = c^v;\ y = c^vt;\ dx = c^vdv;\ dy = c^v(dt + tdv)$$

$$ddy = c^v(ddt + 2dtdv).$$

que

$$(t-1)^2 z + t - lt = a.$$

modo omnes aequationes differentiales, in quibus alterutra variabilis una
es dimensiones nusquam habet, integrari [possunt] seu saltem construibiles
untur. Hac de industria methodo sum usus, quo magis intelligatur, quanti
usus exponentialia in tractandis aequationibus.

17. Aequatio ad quam est perventum haec est

$$(t-1)^2 z + t - lt = a.$$

ulterius reducatur, ut tandem aequatio inter x et y rursus obtineatur;
niam erat $dv = zdt$, erit $z = dv : dt$; quamobrem aequatio abibit in

$$(t-1)^2 dv + tdt - dtlt = adt,$$

vero in

$$dv = \frac{adt - tdt + dtlt}{(t-1)^2}.$$

o denuo integrationem admittit; integrata vero hanc habet formam

$$v = \frac{-a + t - tlt}{t-1}$$

tante vero addita hanc

$$v = \frac{b - a + t - bt - tlt}{t-1}.$$

a vero est $x = c^v$, erit $v = lx$. Et cum sit $y = c^v t$, erit $y = tx$ et ideo $t = y : x$
substitutis habebitur sequens aequatio

$$lx = \frac{bx - ax + y - by - yly + ylx}{y - x}.$$

o oritur haec

$$(b-a)x + (1-b)y = yly - xlx.$$

atur brevitatis causa $b - a = f$ et $1 - b = g$; erit

$$fx + gy = yly - xlx.$$

$$dt^3 + 2tdt^2dv - ttdtdv^2 + ttdv^2 \cdot tdvddt - ttdvddt \cdots$$

Hic cum desit v, ponatur $dv = zdt$, erit ut ante

$$ddt = -zdt^2 - dzdt : z.$$

Exinde reperitur haec aequatio in ordinem reducta:

$$dt + 2tzdt - tdz + ttdz = 0.$$

Quae, cum z unicam tantum habeat dimensionem, separari potest
Cel. Ioh. Bernoulli[1]) in Actis Lips. tradita. Sed sine ulla substitu
eique similes quascumque statim integrare seu ad integralem form
reducere possum, sequenti modo.

16. Reducatur aequatio nostra ad hanc

$$dz + \frac{2zdt}{t-1} + \frac{dt}{tt-t} = 0,$$

ut dz nullo affectum sit coefficiente, tum sumatur id, quo z est affecta
$\frac{2dt}{t-1}$, cuius integrale exprimatur per $2\int\frac{dt}{t-1}$. Iam aequatio proposita
ectur per $c^{2\int\frac{dt}{t-1}}$ et habebitur

$$c^{2\int\frac{dt}{t-1}}dz + \frac{2c^{2\int\frac{dt}{t-1}}zdt}{t-1} + \frac{c^{2\int\frac{dt}{t-1}}dt}{tt-t} = 0.$$

Nunc autem aequatio integrabilis est facta, duorum enim priorum te
integrale est $c^{2\int\frac{dt}{t-1}}z$. Est igitur

$$c^{2\int\frac{dt}{t-1}}z + \int\frac{c^{2\int\frac{dt}{t-1}}dt}{tt-t} = a.$$

Sed cum sit $\int\frac{dt}{t-1} = l(t-1)$, erit

$$c^{2\int\frac{dt}{t-1}} = (t-1)^2.$$

1) Ioh. Bernoulli (1667—1748), *Solutio analytica aequationis* anno 1695, p. 55
Acta erud. 1697, p. 113. *Opera omnia*, t. I, p. 175.

eque

$$(t-1)^2 z + t - lt = a.$$

modo omnes aequationes differentiales, in quibus alterutra variabilis u res dimensiones nusquam habet, integrari [possunt] seu saltem construibil luntur. Hac de industria methodo sum usus, quo magis intelligatur, quan usus exponentialia in tractandis aequationibus.

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$$(t-1)^2 dv + tdt - dtlt = adt,$$

c vero in

$$dv = \frac{adt - tdt + dtlt}{(t-1)^2}.$$

ae denuo integrationem admittit; integrata vero hanc habet formam

$$v = \frac{-a + t - tlt}{t-1}$$

stante vero addita hanc

$$v = \frac{b - a + t - bt - tlt}{t-1}.$$

a vero est $x = c^v$, erit $v = lx$. Et cum sit $y = c^v t$, erit $y = tx$ et ideo $t = y$: substitutis habebitur sequens aequatio

$$lx = \frac{bx - ax + y - by - yly + ylx}{y-x}.$$

lo oritur haec

$$(b-a)x + (1-b)y = yly - xlx.$$

atur brevitatis causa $b - a = f$ et $1 - b = g$; erit

$$fx + gy = yly - xlx.$$

18. Tertium genus aequationum, quarum hic redu
trado, eas complectitur, in quarum singulis terminis alt
eundem tenet dimensionum numerum. Hic duo distin
prout vel ipsius illius variabilis ubique eundem dimension
tiale constans ponitur vel secus. Ad primum casum spec
universalis

$$Px^m dy^{m+2} + Qx^{m-h} dx^h dy^{m-2-h} = dx^m$$

In qua x in singulis terminis m habet dimensiones, et
Significant autem P et Q functiones quascunque ipsius y.
unica substitutione opus est; nempe fiat $x = c^v$, erit

$$dx = c^v dv \text{ et } ddx = c^v (ddv + dv^2) =$$

ergo $ddv = -dv^2$. His subrogatis habetur

$$Pdy^{m+2} + Qdv^h dy^{m+2-h} = dv^m dd$$

postquam nimirum divisa est per c^{mv}.

19. Cum in aequatione inventa v non deprehenda
tuendo loco dv, zdy. Erit

$$ddv = zddy + dydz = -dv^2 = -z^2$$

Hinc inveniotur

$$ddy = -zdy^2 - dydz : z.$$

Substituantur ergo in aequatione inventa loco dv et dd
habebitur haec aequatio

$$Pdy^{m+2} + Qz^h dy^{m+2} = -z^{m+1} dy^{m+2} - z^m$$

Quae divisa per dy^{m+1} abit in hanc

$$Pdy + Qz^h dy = -z^{m+1} dy - z^{m-1}$$

Quae est primi gradus, ut erat propositum. Ad hanc stat
si positum esset

$$x = c^{\int z dy}.$$

hinc

$$ddy = -zdy^2 - dzdy : z.$$

i valores loco x, dx, ddy substituti statim inventam aequationem praeb

20. Alter casus aequationum ad genus tertium pertinentium resp quentem generalem aequationem:

$$Px^m dy^{m+1} + Qx^{m-h} dx^h dy^{m-h-1} = dx^{m-1} ddx.$$

qua aequatione dy ponitur constans, P et Q designant functiones ipsi ascunque. Et ut perspicuum est x in singulis terminis m tenet dimensio onatur, ut ante, $x = c^v$; erit

$$dx = c^v dv \text{ et } ddx = c^v(ddv + dv^2).$$

isce in aequatione substitutis resultat haec aequatio divisione facta per c

$$Pdy^{m+1} + Qdv^h dy^{m-h-1} = dv^{m+1} + dv^{m-1} ddv.$$

aec aequatio ut ulterius reducatur, cum v desit, ponatur $dv = zdy$, erit constans $ddv = dzdy$. Hanc ob rem aequatio ultima transmutabitur

$$Pdy^{m+1} + Qz^h dy^{m+1} = z^{m+1} dy^{m+1} + z^{m-1} dy^m dz.$$

aec autem, si dividatur per dy^m, dabit istam

$$Pdy + Qz^h dy = z^{m+1} dy + z^{m-1} dz.$$

ndet ergo constructio propositae aequationis a constructione huius invent

21. Ex hisce, arbitror, intelligitur, quomodo aequationes differentia undi gradus ad unum aliquod trium expositorum genus pertinentes tract rteat. Facile quidem concedo raro admodum ad tales aequationes pervon quibus non alterutra indeterminata desit; tamen a nemine hoc nom litatem huius inventi impugnatum iri puto. Fieri potest, ut novus aliq mpus aperiatur problemata suggerens, quorum resolutio ad aequationes ta ducat. Memini me aliquando physicum problema quoddam resolvent hanc pervenisse aequationem

$$y^2 ddy = xdxdy.$$

22. Hoc vero praeterea de assumenda constante mono
aequationibus ad secundum genus relatis nihil interest, quod
tiale constans sit assumtum. Potest id esse vel differentiale a
bilis, vel aliud differentiale ex utriusque variabilis differen
compositum, modo id sit, ut natura rei exigit, homogeneum.
generali exemplo locum obtinuit; sed ex illa operatione si
quomodo, si differentiale constans sit qualecunque, aequ
oporteat. Aliter res se habet in duobus reliquis generibus pri
enim necesse est, ut alterutrius variabilis differentiale const
Id si non fuerit, methodo exposita reductio non succedit. Hi
constans debet immutari et aequatio in aliam transforma
utrius variabilis differentiale sit constans.

23. Methodus in hac dissertatione exposita aequatio
secundi gradus ad simpliciter differentiales reducendi consist
stitutione quantitatum exponentialium pro indeterminatis.
latius patet, quam hic est expositum. Possunt eius beneficio
tiones differentiales tertii ordinis ad alias, quae sint tantum
reduci. Et generaliter aequationes differentiales ordinis n ad
quae sint ordinis tantum $n-1$. Aequationum vero cuiusque
tialium, quae hac methodo reducuntur, quoque sunt tria gen
eademque, quae hic sunt exposita. Ex his igitur etiam intel
huiusmodi substitutiones in aequationibus differentialibus p
tandis usum habere possint. Sed de his non opus est plura

CONSTRUCTIO AEQUATIONUM QUARUNDAM DIFFERENTIALIUM QUAE INDETERMINATARU SEPARATIONEM NON ADMITTUNT

Commentatio 11 indicis ENESTROEMIANI
Nova acta eruditorum 1733 p. 369–373

Constructiones, quibus Geometrae ad determinandas quasvis magni
ines utuntur, duplicis sunt generis; ad quorum alterum referri possunt om
onstructiones Geometricae, tam planae, quam solidae et lineares, ad alter
ero pertinent eae constructiones, quae vel quadraturis curvarum, vel re
cationibus perficiuntur. Illas adhibemus in Geometria communi ad radi
equationum algebraicarum quarumcunque exprimendas; id quod efficit
ti constat, intersectione linearum vel rectarum, vel curvarum, prout aequa
blata postulat. Posterioris vero generis constructiones, quas transcender
ppellare licet, inserviunt ad aequationes differentiales resolvendas, quae
gebraicas transmutari nequeunt. Aequationes autem, sive algebraicae, s
ranscendentes, in quibus duae insunt quantitates indeterminatae, huiusm
equirunt constructiones, ut, altera indeterminatarum pro lubitu assum
ltera determinetur; in quo efficiendo pro aequationibus algebraicis, tanqu
ostulatum, praemittitur, ut data magnitudine z, eius quaecunque func
gebraica Z possit exhiberi. Pro differentialibus autem vel transcendenti
equationibus insuper requiritur, ut, posita quantitate z functio eius quaec
ue transcendens $\int Zdz$, in qua Z significat functionem quamcunque ipsiu
ve algebraicam, sive transcendentem, denuo definiri, atque adeo tanqu
ata considerari possit. Hanc ob rem igitur, quoties aequatio proposita
otest transformari, ut altera indeterminata, vel eius quaedam functio, aeq

aequationis constructio erit in promtu. Vocari autem sc
transmutatio indeterminatarum separatio; ex quo sim
semper ad aequationes transcendentes construendas hui
sollicite requiratur. In algebraicis quidem aequationibu
est necessaria ad constructionem adornandam. Quomo
indeterminatae sint permixtae, totum negotium aeque fa
quod ad differentiales aequationes attinet, ne unica qui
quae construi, neque tamen separari, queat. Usitatae
omnes ita sunt comparatae, ut ex iis ipsis separatio indet
alias fuerit inventu difficillima, sponte sequatur. Hanc c
praestitisse arbitror, cum nuper in constructiones aeq
differentialium, quae indeterminatas a se invicem separa
dissem, simulque cognovissem, has contructiones plus
ante concedi solere observaveram. Prima aequatio, quae
formae[1]):

$$dy + \frac{y^2 dx}{x} = \frac{x dx}{x^2 - 1},$$

in qua non solum indeterminatas a se invicem separare
ipsa etiam constructio demonstrabit, huiusmodi separ
non posse. Si enim succederet, perspicuum erit, compar
ellipsium dissimilium ex ea esse secuturam, quae tam
concessa quadratura, exhiberi potest. Istam vero aequa
construo.

Fiant super eodem axe coniugato infinitae ellipse
transverso a se invicem discrepant. Ex his conficiatur
abscissae aequales capiantur axibus ellipsium transv
aequales peripheriis earundem ellipsium. Hoc facto, voce
constans 1, abscissa huius novae curvae, seu axis transv
ponatur $= r$, et applicata, seu perimeter eiusdem ell
nunc $x = \sqrt{(r^2 - 1)}$, eritque $y = \frac{(r^2 - 1)\, dq}{q r dr}$, quae c
data r per rectificationem curvae cognitae habetur,
Simili modo deductus sum mox ad constructionem cele

1) Vide L. Euleri Commentationem 28 indicis Enestroemiani
aequationum differentialium sine indeterminatarum separatione, Comment
1738, p. 168; *Leonhardi Euleri Opera omnia*, series 1, vol. 20 p. 1.

m tradidit. Deinceps quidem variae comparuerunt meditationes, quae au
unes nihil aliud continent, nisi ut casus particulares, seu valores loco n s
itnendos, exhibuerint, quibus ista aequatio separationem et integratior
noque admittit. Nemo vero, quantum scio, ne unicum quidem assigna
sum, quo constructio perfici possit, praeter illos exhibitos. Ut taceam ig
niversalem, quicquid n significet, constructionem, quae, nisi meae meth
neficio, vix a quoquam poterit inveniri: sequenti ratione ego istam aeq
onem resolvo[2]). Quantitas ista differentialis[3])

$$n(n+4)dz(1-z^2)^{\frac{-n-4}{2n+4}} + 2dz(1-z^2)^{\frac{-n-1}{2n+4}}\left(e^{\frac{2z\sqrt{f}}{n+2}} + e^{\frac{-2z\sqrt{f}}{n+2}}\right)$$

qua z est variabilis, f constans, et e numerus, cuius logarithmus hyperbol
t 1, ita integretur, ut, facto $z=0$, tota evanescat. Quod quidem integr
iamsi re ipsa exhiberi nequeat, tamen per quadraturas construi, ideo
uquam cognitum considerari poterit. In hoc deinceps integrali ponatur z
habebitur quantitas, quae erit functio quaedam ipsius f. Scribatur p
hac functione ax^{n+2} loco f, et quantitas resultans, quae tota ex x et const
bus erit composita, vocetur P. Invento nunc hoc modo P, dico,
$=\frac{dP}{Pdx}$, qui est verus ipsius y valor in aequatione proposita

$$ax^n dx = dy + y^2 dx.$$

1) Iacopo Riccati (1676–1754) primus quidem proposuit problema casus separabilis
endi, *Acta erud.*, Suppl. t. VIII (1723/4), p. 66 et *Acta erud.* 1723, p. 509, sed Dan. Berno
700–1782) primus hos casus publici iuris fecit, *Acta erud.* 1725 p. 473. Vide Commentatione
, 70, 95, 269, 284 huius voluminis. Vide quoque *Institutionum calculi integralis* vol. I, § 436 –
l. II, § 831–841, 904, 929–966. Vide porro L. Euleri Commentationes 431, 595, 678,
onstructio aequationis differentio-differentialis

$$(a+bx)\,ddz + (c+ex)\frac{dxdz}{x} + (f+gx)\frac{zdx^2}{xx} = 0$$

mto elemento dx constante. *Novi comment. acad. sc. Petrop.* 17, 1773, p. 125. *Summatio frac
ntinuae, cuius indices progressionem arithmeticam constituunt, dum numeratores omnes sunt uni
i simul resolutio aequationis Riccatianae per huiusmodi fractiones docetur.* *Opuscula anal.* 2, 1785, p
*ethodus nova investigandi omnes casus, quibus hanc aequationem differentialem ddy (1 —
$bx dx dy - cy dx^2 = 0$ resolvere licet.* *Institutiones calculi integralis* 4, 1794, p. 533. *Ana
cilis aequationem Riccatianam per fractionem continuam resolvendi. *Mém. Pétersb.* 6, 1818, p
eonhardi Euleri *Opera omnia*, series I, vol. 11, 12, 23. H.

2) Vide L. Euleri Commentationem 31, p. 21 huius voluminis. H.

3) Ponendo n loco z et $\frac{n+1}{n+2} = k + \frac{1}{2}$, haec formula eadem est formula ac in Commentatio
cripta § 17. Vide p. 34, vide quoque notam 1. H.

Notandum est autem, hanc solutionem locum ...
numerus intra hos terminos 0 et -2 contentus. At huic ...
remedium adhibetur, ita, ut ista constructio nihilominus ...
habenda. Cum enim, uti constat ex iis, quae Cl. DANIEL B...
aequatione in publicam edidit, ista aequatio, si sit separab...
separari quoque possit in casu $n = \frac{-m}{m+1}$ vel $n = -m - \ldots$
casus omnes intra limites 0 et -2 contentos reduci posse ...
limites -2 et -4 comprehenduntur, et hanc ob rem non au...
Observo autem, formulam illam differentialem[1])

$$n(n+4)dz(1-z^2)^{\frac{-n-4}{2n+4}} + 2dz(1-z^2)^{\frac{-n-4}{2n+4}}\left(c^{\frac{2z\sqrt{\ldots}}{n+2}}\ldots\right.$$

quoties $\frac{-n-4}{2n+4}$ sit vel 0 vel numerus integer affirmativus, ...
integrari. Hoc vero accidit, quoties fuerit $n = \frac{-4k}{2k-1}$, do...
quemcumque affirmativum integrum. Quia deinde aequatio...
x est $\frac{-n}{n+1}$, ad hanc $ax^n dx = dy + y^2 dx$ potest reduci...
quoque integrabilis, si fuerit $n = \frac{-4k}{2k+1}$.

Atque sic prodeunt illi ipsi casus, iam ab aliis eruti, qu...
in aequatione proposita a se invicem possunt separari.

1) Vide notam 3 p. 17.

CONSTRUCTIO AEQUATIONIS DIFFERENTIALI
$ax^n dx = dy + y^2 dx$

Commentatio 31 indicis ENESTROEMIANI
Commentarii academiae scientiarum Petropolitanae 6 (1732/3), 1738, p. 124—137

SUMMARIUM

Ex manuscriptis academiae scientiarum Petropolitanae nunc primum editum.

Maxime agitata est inter Geometras ista aequatio ab illustri Comite Riccati pri
roposita. Nemo vero eius constructionem, nisi pro certis litterae n valoribus, d
'anto ergo magis facienda est methodus ab Eulero hic proposita cuius beneficio o
uius rei difficultates superavit, atque universalem huius aequationis constructio
edit.

1. Communicavi nuper cum Societate[1]) specimen constructionis ae
ionis cuiusdam differentialis, in qua non solum indeterminatas a se invi
eparare non potueram, sed etiam monstraveram ex ipsa constructione hu
nodi separationem omnino non posse exhiberi. Differt quidem meus ibi d
onstruendi modus ab usitatis: attamen iis nequaquam illum esse postpo
lum quilibet intelliget, qui hanc schedam inspexerit. Neque vero tum temp
hanc methodum ulterius extendere, atque ad alias aequationes accommo
icuit, quia ex posita constructione ad aequationem demum perveneram,
utem vicissim data aequatione constructionem eruere potueram. At dein
um hanc rem diligentius contemplatus essem, voti mei compos quodamm
um factus, ita ut hanc methodum invertere, atque propositae aequati
onstructionem invenire potuerim.

1) Vide notam p. 16. H.

quam Clar. COMES RICCATI[1]) primum Geometris examinand
vero eius constructionem, nisi pro certis litterae n valorib
methodi beneficio omnes difficultates feliciter superavi,
huius aequationis constructionem inveni, in qua nihil omn
Non solum autem unicam haec methodus suppeditat
plures, immo etiam innumerabiles. Merito igitur mihi
tantam praestantiam adscribere, ut ad omnes aequation
struendas, in quibus aliae methodi frustra sunt adhibita
stratura.

3. Quemadmodum in superiore Dissertatione[2]) arcu
ad constructionem huius aequationis

$$dy + \frac{y^2 dx}{x} = \frac{x dx}{x^2 - 1},$$

ita pro aequatione proposita alia opus erit curva, loco
Quam ut inveniam pono universalissime eius elementu
P et R sunt functiones ipsius z tales, quae iisdem factis o
in elemento elliptico, deducant ad aequationem proposi
series quaedam in considerationem veniat,

$$R = 1 + AgQ + ABg^2Q^2 + ABCg^3Q^3 + ABC$$

in qua serie est Q functio quaedam ipsius z, g linea data
curvae, A, B, C, D, etc. coefficientes constantes. Pona

$$PRdz = dZ;$$

erit ergo

$$Z = \int Pdz + \int AgPQdz + \int ABg^2PQ^2dz + \int ABC$$

4. Ita autem P et Q a se invicem pendeant, ut c
possint ad $\int Pdz$ reduci. Sit ergo

1) Vide p. 17 et notam 1 p. 17.
2) Vide notam p. 16.

$$\int PQ^3 dz = \alpha\beta\gamma \int P dz + O_3 \text{ etc.}$$

enotant hic O_1, O_2, O_3 etc. quantitates algebraicas. Post peractam hoc m tegrationem ponatur $z = h$; est autem h talis quantitas, quae loco z s tuta faciat omnes eas quantitates algebraicas O_1, O_2, O_3 etc. evanescere, at m fiat $\int P dz = H$, quantitati prorsus constanti. Ex his igitur, facto p tegrationem $z = h$, erit

$$Z = H(1 + A\alpha g + AB\alpha\beta g^2 + ABC\alpha\beta\gamma g^3 + \text{etc.})$$

cta iam parametro g variabili obtinebuntur infiniti valores ipsius Z initis ipsius g, atque ex dato elemento $PRdz$ poterit construi curva, in q abscissae designentur littera g, applicatae sunt $= Z$.

5. Hoc itaque modo poterit construi summa seriei

$$1 + A\alpha g + AB\alpha\beta g^2 + ABC\alpha\beta\gamma g^3 + \text{etc.}$$

uamvis forte ex sui ipsius consideratione summa prorsus non possit de nari. Utor autem ad summam huius seriei investigandam methodo u mmae serierum inventionem ad resolutionem aequationum reducendi, qu nno praeterito exposui[1]), ut nanciscar aequationem, cuius resolutio a se ius summa pendeat. Perspicuum enim est, utcunque haec aequatio result erit perplexa, eius tamen constructionem in promtu futuram. Nunc ig hil aliud est faciendum, nisi ut quantitates A, B, C etc. et α, β, γ ficiantur eiusmodi, ut summae seriei istius inventio ad resolutionem hu quationis

$$ax^n dx = dy + y^2 dx$$

ducatur. Hoc vero loco id est efficiendum, ut series

$$1 + AgQ + ABg^2Q^2 + ABCg^3Q^3 + \text{etc.}$$

ossit in summam redigi, quia alias valor ipsius R non esset cognitus, et proi tegra constructio inutilis. Quamobrem non licebit loco A, B, C etc. val uosvis pro arbitrio accipere, sed tales, quae hanc seriem summabilem redd

1) L. Euleri Commentatio 25: *Methodus generalis summandi progressiones.* Comment. Petrop. 6, 1738, p. 68. Vide quoque *Institutionum calculi differentialis* p. 238. Leonhardi Eu era omnia, series I, vol. 14 et 20. H.

ut eius summatio perducatur ad resolutionem aequationis

$$ax^n dx = dy + y^2 dx;$$

hanc ipsam aequationem in seriem resolvo. Quod ut commodius
pono[1])

$$y = \frac{dt}{tdx},$$

sumtoque dx constante erit

$$ax^n dx = \frac{ddt}{tdx} \text{ seu } ax^n t dx^2 = ddt.$$

Nunc more consueto substituo loco t hanc seriem

$$1 + \mathfrak{A}x^{n+2} + \mathfrak{B}x^{2n+4} + \mathfrak{C}x^{3n+6} + \text{etc.},$$

erit

$$ddt = (n+1)(n+2)\mathfrak{A}x^n dx^2 + (2n+3)(2n+4)\mathfrak{B}x^{2n}$$
$$(3n+5)(3n+6)\mathfrak{C}x^{3n+4}dx^2 + \text{etc.}$$

Huic igitur seriei aequalis esse debet $ax^n t dx^2$, seu ista series

$$ax^n dx^2 + \mathfrak{A}ax^{2n+2}dx^2 + \mathfrak{B}ax^{3n+4}dx^2 + \text{etc.};$$

propterea aequales facio terminos homogeneos determinandis litte
pro arbitrio assumtis, fietque

$$\mathfrak{A} = \frac{a}{(n+1)(n+2)},\ \mathfrak{B} = \frac{\mathfrak{A}a}{(2n+3)(2n+4)},\ \mathfrak{C} = \frac{\mathfrak{B}a}{(3n+5)(3n}$$

Ponatur $ax^{n+2} = f$ brevitatis gratia, erit

$$t = 1 + \frac{f}{(n+1)(n+2)} + \frac{f^2}{(n+1)(n+2)(2n+3)(2n+}$$
$$\frac{f^3}{(n+1)(n+2)(2n+3)(2n+4)(3n+5)(3n+6)} + \text{et}$$

Huius ergo seriei summatio pendet a constructione aequationi

$$ax^n dx = dy + y^2 dx.$$

1) Vide *Institutionum calculi integralis* vol. II § 955, 1068—1080; *Opera*
253 ff. Vide quoque p. 12 et notam 2 p. 3.

possit transmutari, habebitur simul constructio aequationis propositae.

Sed quo haec series, quippe quae nimis est generalis, aliquanto magis gatur, et determinatio litterarum arbitrariarum facilior efficiatur, pono mula $PRdz$ initio assumta

$$P = \frac{1}{(1+bz^\mu)^\nu} \text{ et } Q = \frac{z^\mu}{1+bz^\mu}.$$

rgo

$$= \int \frac{dz}{(1+bz^\mu)^\nu}, \int PQdz = \int \frac{z^\mu dz}{(1+bz^\mu)^{\nu+1}} \text{ et } \int PQ^2 dz = \int \frac{z^{2\mu} dz}{(1+bz^\mu)^{\nu+2}} \text{ etc.}$$

nt autem haec omnia integralia ad primum $\int \frac{dz}{(1+bz^\mu)^\nu}$ reduci; est generaliter

$$\frac{z^{\theta\mu} dz}{(1+bz^\mu)^{\nu+\theta}} = \frac{(\theta-1)\mu+1}{b\mu(\nu+\theta-1)} \int \frac{z^{(\theta-1)\mu} dz}{(1+bz^\mu)^{\nu+\theta-1}} - \frac{1}{b\mu(\nu+\theta-1)} \cdot \frac{z^{(\theta-1)\mu+1}}{(1+bz^\mu)^{\nu+\theta-1}}.$$

ob rem erit

$$\int \frac{z^\mu dz}{(1+bz^\mu)^{\nu+1}} = \frac{1}{b\mu\nu} \int \frac{dz}{(1+bz^\mu)^\nu} - \frac{1\cdot z}{b\mu\nu(1+bz^\mu)^\nu},$$

$$\int \frac{z^{2\mu} dz}{(1+bz^\mu)^{\nu+2}} =$$

$$\frac{(\mu+1)}{\ldots(\nu+1)} \int \frac{dz}{(1+bz^\mu)^\nu} - \frac{(\mu+1)z}{b^2\mu^2\nu(\nu+1)(1+bz^\mu)^\nu} - \frac{1\cdot z^{\mu+1}}{b\mu(\nu+1)(1+bz^\mu)^{\nu+1}} \text{ etc.}$$

it ergo h eiusmodi esse quantitas, ut loco z substituta [§ 4] faciat

$$\frac{z^{\mu\theta+1}}{(1+bz^\mu)^{\nu+\theta}} = 0.$$

ero poterit esse $h = 0$, quia tum plerumque simul quantitas $\int \frac{dz}{(1+bz^\mu)^\nu}$ sceret. Comparatis iam his reductionibus cum supra assumtis, detertur litterae α, β, γ, δ etc. Erit scilicet

$$\alpha = \frac{1}{b\mu\nu}, \beta = \frac{\mu+1}{b\mu(\nu+1)}, \gamma = \frac{2\mu+1}{b\mu(\nu+2)} \text{ etc.}$$

factum ex duobus factoribus, habere oportere. Quo aute

$$1 + AgQ + ABg^2Q^2 + ABCg^3Q^3 + \text{e}$$

possit summari, facio

$$A = \frac{1}{\pi(\pi + \varrho)}, \ B = \frac{1}{(\pi + 2\varrho)(\pi + 3\varrho)}, \ C = \frac{}{(\pi + 4\varrho)}$$

atque tum series ope methodi meae universalis serie
summari. Pono primo brevitatis gratia $gQ = q^2$, erit

$$R = 1 + \frac{q^2}{\pi(\pi + \varrho)} + \frac{q^4}{\pi(\pi + \varrho)(\pi + 2\varrho)(\pi + 3\varrho)}$$

facioque $R - 1 = S$, erit

$$S = \frac{q^2}{\pi(\pi + \varrho)} + \frac{q^4}{\pi(\pi + \varrho)(\pi + 2\varrho)(\pi + 3\varrho)}$$

Multiplico nunc ubique per $\varrho q^{\frac{\pi-\varrho}{\varrho}}$ sumoque differentialia, e

$$\frac{\varrho d(q^{\frac{\pi-\varrho}{\varrho}} S)}{dq} = \frac{q^{\frac{\pi}{\varrho}}}{\pi} + \frac{q^{\frac{\pi+2\varrho}{\varrho}}}{\pi(\pi + \varrho)(\pi + 2\varrho)} +$$

Iam per ϱ multiplico sumoque denuo differentialia po
prodibit

$$\frac{\varrho^2 dd(q^{\frac{\pi-\varrho}{\varrho}} S)}{dq^2} = q^{\frac{\pi-\varrho}{\varrho}} + \frac{q^{\frac{\pi+\varrho}{\varrho}}}{\pi(\pi + \varrho)} + \text{etc.}$$

$$= q^{\frac{\pi-\varrho}{\varrho}} + q^{\frac{\pi-\varrho}{\varrho}} \left(\frac{q^2}{\pi(\pi + \varrho)} + \frac{q^4}{\pi(\pi + \varrho)(\pi + 2\varrho)(\pi +}\right.$$

Habebimus ergo restituto S loco seriei

$$\frac{q^2}{\pi(\pi + \varrho)} + \text{etc.}$$

hanc aequationem

$$\varrho^2 dd(q^{\frac{\pi-\varrho}{\varrho}} S) = q^{\frac{\pi-\varrho}{\varrho}} dq^2 + q^{\frac{\pi-\varrho}{\varrho}} S dq^2$$

$$\varrho^2 ddT = q^{\frac{n-\varrho}{\varrho}} dq^2 + Tdq^2.$$

9. Ad hanc aequationem integrandam pono $T = rs$, erit

$$ddT = rdds + 2\,drds + sddr,$$

quibus substitutis habetur

$$\varrho^2 rdds + 2\,\varrho^2 drds + \varrho^2 sddr = q^{\frac{n-\varrho}{\varrho}} dq^2 + rsdq^2,$$

quae in duas aequationes discerpatur,

$$\varrho^2 rdds = rsdq^2,$$

et

$$2\,\varrho^2 drds + \varrho^2 sddr = q^{\frac{n-\varrho}{\varrho}} dq^2.$$

Harum prior per r divisa abit in hanc $\varrho^2 dds = sdq^2$, quae per ds multiplicata dat hanc

$$\varrho^2 dsdds = sdsdq^2,$$

cuius integralis est

$$\varrho^2 ds^2 = s^2 dq^2,$$

seu haec $\varrho ds = sdq$, quae denuo integrata dat

$$\varrho ls = q \text{ atque } s = c^{\frac{q}{\varrho}}$$

denotante c numerum, cuius logarithmus est 1. Invento itaque s assumatur alteram aequationem

$$2\,\varrho^2 drds + \varrho^2 sddr = q^{\frac{n-\varrho}{\varrho}} dq^2,$$

quae substituto loco s valore invento $c^{\frac{q}{\varrho}}$ abit in istam

$$2\,\varrho c^{\frac{q}{\varrho}} dqdr + \varrho^2 c^{\frac{q}{\varrho}} ddr = q^{\frac{n-\varrho}{\varrho}} dq^2.$$

Ponatur

$$dr = vdq, \text{ erit } ddr = dvdq$$

$$2\varrho c^{\frac{q}{\varrho}} v\,dq + \varrho^2 c^{\frac{q}{\varrho}} dv = q^{\frac{\pi-\varrho}{\varrho}}\,dq,$$

quam multiplico per $c^{\frac{q}{\varrho}}$, ut prodeat

$$2\varrho c^{\frac{2q}{\varrho}} v\,dq + \varrho^2 c^{\frac{2q}{\varrho}} dv = c^{\frac{q}{\varrho}} q^{\frac{\pi-\varrho}{\varrho}}\,dq,$$

cuius integralis est

$$\varrho^2 c^{\frac{2q}{\varrho}} v = \int c^{\frac{q}{\varrho}} q^{\frac{\pi-\varrho}{\varrho}}\,dq.$$

Fit igitur

$$v = \frac{1}{\varrho^2} c^{\frac{-2q}{\varrho}} \int c^{\frac{q}{\varrho}} q^{\frac{\pi-\varrho}{\varrho}}\,dq,$$

et

$$\int v\,dq \quad \text{seu} \quad r = \frac{1}{\varrho^2}\int c^{\frac{-2q}{\varrho}}\,dq \int c^{\frac{q}{\varrho}} q^{\frac{\pi-\varrho}{\varrho}}\,dq.$$

Erit ergo

$$rs = T = \frac{1}{\varrho^2} c^{\frac{q}{\varrho}} \int c^{\frac{-2q}{\varrho}}\,dq \int c^{\frac{q}{\varrho}} q^{\frac{\pi-\varrho}{\varrho}}\,dq$$

et

$$S = \frac{1}{\varrho^2} c^{\frac{q}{\varrho}} q^{\frac{\varrho-\pi}{\varrho}} \int c^{\frac{-2q}{\varrho}}\,dq \int c^{\frac{q}{\varrho}} q^{\frac{\pi-\varrho}{\varrho}}\,dq.$$

10. Quoniam in hac forma inventa duplex involvitur inte
dum est eas ita institui debere, ut tam S quam $\frac{dS}{dq}$ fiant $=0$, posit
admodum ex serie, cui S est aequale, apparet. His observatis ha

$$R = 1 + \frac{1}{\varrho^2} c^{\frac{q}{\varrho}} q^{\frac{\varrho-\pi}{\varrho}} \int c^{\frac{-2q}{\varrho}}\,dq \int c^{\frac{q}{\varrho}} q^{\frac{\pi-\varrho}{\varrho}}\,dq.$$

Est vero $q = \sqrt{g}Q$, atque ob

$$Q = \frac{z^\mu}{1+bz^\mu}, \text{ erit } q = \sqrt{\frac{gz^\mu}{1+bz^\mu}}.$$

Dabitur igitur ex his $\int PR\,dz$ seu

$$\int \frac{R\,dz}{(1+bz^\mu)^\nu}.$$

Quare si litteris π, ϱ, μ et ν tribuantur debiti valores in n, in prom
tionis propositae

$$ax^n dx = dy + y^2 dx$$

constructio.

quae positis loco A, α, B, β, C, γ, etc. electis valoribus transmutatur in

$$1 + \frac{g}{b\mu\nu\pi(\pi+\varrho)} + \frac{(\mu+1)g^2}{b^2\mu^2\nu(\nu+1)\pi(\pi+\varrho)(\pi+2\varrho)(\pi+3\varrho)} + \text{etc.},$$

cuius haec est lex, ut terminus indicis $\theta+1$ divisus per terminum indici

$$= \frac{g(1+(\theta-1)\mu)}{b\mu(\nu+\theta-1)(\pi+(2\theta-2)\varrho)(\pi+(2\theta-1)\varrho)}.$$

In serie vero, quam § 6 ex aequatione proposita elicuimus, est similis q termini indicis $\theta+1$ per terminum indicis θ divisi

$$= \frac{f}{(\theta n+2\theta-1)(\theta n+2\theta)}.$$

Quo igitur hae duae series congruant, oportet ut hi duo quoti sint in aequales. Fiat ergo primo

$$\frac{g}{b} = f \text{ seu } g = bf,$$

hoc posito debebit esse

$$\frac{1}{(\theta n+2\theta-1)(\theta n+2\theta)} = \frac{\theta\mu-\mu+1}{(\mu\nu+\mu\theta-\mu)(\pi+2\theta\varrho-2\varrho)(\pi+2\theta\varrho-\varrho)}$$

Unde si aequatio secundum dimensiones ipsius θ ordinetur, et coeffici cuiusque ipsius θ potentiae ponatur $=0$, prodibunt quatuor aequat ex quibus μ, ν, π, et ϱ determinabuntur in n. Neque vero unica datur so sed sunt quatuor diversae quae ad nostrum institutum pertinent.

Prima dat $\mu = \frac{2n+4}{3n+4}$, $\nu = 1$, $\pi = n+1$ et $\varrho = \frac{n+2}{2}$.

Secunda dat $\mu = \frac{2n+4}{n}$, $\nu = 1$, $\pi = \frac{n}{2}$ et $\varrho = \frac{n+2}{2}$.

Tertia dat $\mu = 2$, $\nu = \frac{n+1}{n+2}$, $\pi = \frac{n+2}{2}$ et $\varrho = \frac{n+2}{2}$.

Quarta dat[1] $\mu = \frac{2}{3}$, $\nu = \frac{n+1}{n+2}$, $\pi = n+2$ et $\varrho = \frac{n+2}{2}$.

---

1) Editio princeps: $\mu = \frac{1}{3}$, $\pi = (n+2)\sqrt{2}$, $\varrho = \frac{n+2}{\sqrt{2}}$. Correxit I

$$\frac{z^{\mu\theta+1}}{(1+bz^\mu)^{\nu+\theta}}$$

evanescere debeat facto $z = h$. Fit hoc quidem si $z = 0$, se
alius requiratur, facile apparet, id non evenire posse, nisi
quolibet igitur casu ipsius n talis eligenda est solutio, ut

$$\frac{z^{\mu\theta+1}}{(1+bz^\mu)^{\nu+\theta}}$$

fiat $= 0$ posito $z = \infty$. Denotat hic autem θ numerum qu
affirmativum non excepta cyphra, quamobrem et ν nu
numerus negativus, quia alioquin binomium $1 + bz^\mu$ in n
At μ tam affirmativum quam negativum numerum signifi
duplex existit huius rei consideratio, prout fuerit μ vel af
vel negativus. Sit primo μ numerus affirmativus $= + \lambda$,

$$\frac{z^{\lambda\theta+1}}{(1+bz^\lambda)^{\nu+\theta}}$$

fiat $= 0$, posito $z = \infty$, oportere maximum ipsius z exp
natore, qui est $\lambda\nu + \lambda\theta$, maiorem esse eiusdem z exponent
est $\lambda\theta + 1$. Erit igitur $\lambda\nu > 1$. Sin autem fuerit μ nu
$= -\lambda$, fiet

$$\frac{z^{-\lambda\theta+1}}{(1+bz^{-\lambda})^{\nu+\theta}} = \frac{z^{\lambda\nu+1}}{(z^\lambda+b)^{\nu+\theta}},$$

quae quantitas ut fiat $= 0$ posito $z = \infty$, debebit ess

$$\lambda\nu + \lambda\theta > \lambda\nu + 1, \text{ seu } \lambda\theta > 1,$$

idquod in casu $\theta = 0$ fieri nequit. Quocirca μ nunquam
negativus. In prima igitur solutione, quia est $\nu = 1$,
$\frac{2n+4}{3n+4}$ numerus positivus, toties simul esse debebit nu

excipiuntur igitur ii casus, quibus $\frac{2n+4}{3n+4}$ est 1 vel unitat

contineatur intra hos limites 0 et $-\frac{4}{3}$, prima solutio adhib

solutione, quia iterum est $\nu = 1$, similiter excipiuntur casu

st unitas seu unitate minor. Semper igitur haec solutio locum habebit
autum exceptis casibus, quando n continetur intra hos limites -4 et 0.
ertia solutione, quia μ iam est numerus positivus nempe $= 2$, debebit tam
$\frac{n+2}{n+2}$ esse numerus unitate maior. Hac igitur semper uti poterimus, n
ontineatur intra hos limites -2 et 0; quoties ergo secunda locum habet, t
t tertia poterit usurpari. In quarta denique solutione, quia μ quoque est num
ffirmativus, scilicet $\frac{2}{3}$[1]), requiritur, ut $\frac{2n+2}{3n+6}$ sit numerus unitate maio
uod accidit, quoties n continetur intra hos limites -2 et -4. In his ig
asibus quarta solutione uti conveniet. Ex quibus invicem comparatis
icitur, semper hoc modo aequationis propositae constructionem exhi
osse, nisi n contineatur intra hos angustos limites $-\frac{4}{3}$ et -2.

13. Quo autem totum hoc negotium evidentius percipiatur, accom
abo, quae hactenus tradita sunt, ad casum particularem, quo est $n = 2$
aque construenda sit haec aequatio

$$ax^2dx = dy + y^2dx.$$

ro hoc casu eligo solutionem tertiam, eritque propterea

$$\mu = 2,\ \nu = \frac{3}{4},\ \pi = \varrho = 2.$$

is valoribus substitutis habebitur

$$S = \frac{1}{4}c^{\frac{q}{2}}\int c^{-q}dq\int c^{\frac{q}{2}}dq.$$

$$\text{Est vero } \int c^{\frac{q}{2}}dq = 2c^{\frac{q}{2}} + i, \text{ ergo}^{2)}$$

$$\int c^{-q}dq\int c^{\frac{q}{2}}dq = \int 2c^{-\frac{q}{2}}dq + i\int c^{-q}dq = -c^{-\frac{q}{2}} - ic^{-q} + k.$$

1) Editio princeps: $\frac{1}{3}$ loco $\frac{2}{3}$, $\frac{n+4}{3n+6}$ loco $\frac{2n+2}{3n+6}$ et infra $-\frac{5}{2}$ loco 4. Correxit H. I

2) Cuius formulae posterius membrum emendare oportet. Habebitur

$$-4c^{-\frac{q}{2}} - ic^{-q} + k \qquad \text{et in formulis sequent}$$

$$S = \frac{k}{4}c^{\frac{q}{2}} - \frac{i}{4}c^{-\frac{q}{2}} - 1, \qquad k = 4 + i, \qquad i = -2, \qquad k = 2$$

$$S = \frac{c^{\frac{q}{2}} + c^{-\frac{q}{2}}}{2} - 1 \qquad\qquad R = \frac{c^{\frac{q}{2}} + c^{-\frac{q}{2}}}{2}.$$

$$\int PRdz = \frac{1}{2}\int \frac{dz\left(c^{\frac{1}{2}\sqrt{\frac{bfz^2}{1+bz^2}}} + c^{-\frac{1}{2}\sqrt{\frac{bfz^2}{1+bz^2}}}\right)}{(1+bz^2)^{\frac{3}{4}}}$$

Correxit H. I

Consequenter prodit

$$S = \frac{k}{4}c^{\frac{q}{2}} - \frac{i}{4}c^{\frac{-q}{2}} - \frac{1}{4}.$$

Quia iam posito $q = 0$ debet evanescere S, habebitur ista aeq

$$\frac{k}{4} - \frac{i}{4} - \frac{1}{4} = 0, \text{ seu } k = 1 + i.$$

Porro cum $\frac{dS}{dq}$ debeat esse $= 0$, si $q = 0$, proveniet $i + k = 0$.

$$dS = \frac{k}{8}c^{\frac{q}{2}}dq + \frac{i}{8}c^{\frac{-q}{2}}dq,$$

et ideirco facto $q = 0$, sit

$$\frac{dS}{dq} = \frac{k}{8} + \frac{i}{8} = 0.$$

Ex his igitur conditionibus invenitur $i = -\frac{1}{2}$, et $k = \frac{1}{2}$; quan

$$S = \frac{c^{\frac{q}{2}} + c^{\frac{-q}{2}}}{8} - \frac{1}{4}, \text{ atque } R = \frac{3}{4} + \frac{c^{\frac{q}{2}} + c^{\frac{-q}{2}}}{8}.$$

Quoniam vero est $\mu = 2$ et $g = bf$, erit

$$q = \sqrt{\frac{bfz^2}{1 + bz^2}}, \text{ adeoque } R = \frac{3}{4} + \frac{1}{8}c^{\frac{1}{2}\sqrt{\frac{bfz^2}{1+bz^2}}} + \frac{1}{8}c^{-\frac{1}{2}\sqrt{}}$$

Consequenter reperitur

$$\int PRdz = \frac{3}{4}\int\frac{dz}{(1 + bz^2)^{\frac{3}{2}}} + \frac{1}{8}\int\frac{dz\left(c^{\frac{1}{2}\sqrt{\frac{bfz^2}{1+bz^2}}} + c^{-\frac{1}{2}\sqrt{\frac{bf}{1+}}}\right.}{(1 + bz^2)^{\frac{3}{2}}}$$

Quod integrale ita capiatur, ut posito $z = 0$ ipsum fiat $= 0$, quo $z = \infty$, et prodibit quantitas, quae ut functio ipsius f potest deinde f variabilis, eiusque loco ponatur ax^4, erit ista functio per (vide §6). Atque invento hoc t erit $y = \frac{dt}{tdx}$, qui est verus valo aequatione proposita

$$ax^2dx = dy + y^2dx.$$

$$= ady + y\,dx,$$

modo n non continuatur intra hos limites 0 et − 2. Uti enim poterimu
tione tertia, in qua sit

$$\mu = 2,\ \nu = \frac{n+1}{n+2},\ \pi = \varrho = \frac{n+2}{2}.$$

igitur

$$S = \frac{1}{\varrho^2} c^{\frac{q}{\varrho}} \int c^{\frac{-2q}{\varrho}} dq \int c^{\frac{q}{\varrho}} dq.$$

gratione simili quo supra modo instituta, reperitur[1]

$$S = \frac{k}{\varrho^2} c^{\frac{q}{\varrho}} - \frac{i}{\varrho^2} c^{\frac{-q}{\varrho}} - \frac{1}{\varrho^2},$$

et k ex his aequationibus debent definiri $k = 1 + i$, et $k + i = 0$, est erg
nte $i = -\frac{1}{2}$ et $k = \frac{1}{2}$. Quapropter est

$$S = \frac{1}{2\varrho^2} c^{\frac{q}{\varrho}} + \frac{1}{2\varrho^2} c^{\frac{-q}{\varrho}} - \frac{1}{\varrho^2} \text{ atque } R = 1 - \frac{1}{\varrho^2} + \frac{1}{2\varrho^2} c^{\frac{q}{\varrho}} + \frac{1}{2\varrho^2} c^{\frac{-q}{\varrho}}$$

posito loco ϱ valore $\frac{n+2}{2}$ habebitur

$$R = \frac{n(n+4) + 2\,c^{\frac{2q}{n+2}} + 2\,c^{\frac{-2q}{n+2}}}{(n+2)^2}.$$

vero ut ante

$$q = \sqrt{\frac{bfz^2}{1+bz^2}},\ \text{at } Pdz = \frac{dz}{(1+bz^2)^{\frac{n+1}{n+2}}}$$

1) In hac formula et in formulis sequentibus ϱ^2 supprimendum est. Vide notam p. 20.
bitur

$$S = \frac{c^{\frac{q}{\varrho}} + c^{-\frac{q}{\varrho}}}{2} - 1, \qquad R = \frac{c^{\frac{q}{\varrho}} + c^{-\frac{q}{\varrho}}}{2}, \qquad R = \frac{c^{\frac{2q}{n+2}} + c^{\frac{-2q}{n+2}}}{2}$$

$$\int PRdz = \int \frac{dz}{2(1+bz^2)^{\frac{n+1}{n+2}}} \left(c^{\frac{2q}{n+2}} + c^{\frac{-2q}{n+2}}\right)$$

Correxit H. D.

$$\int PRdz = \frac{\quad}{(n+2)^2}\int \frac{\quad}{(1+bz^2)^{\frac{n+1}{n+2}}} (n(n \ldots$$

ubi loco $\sqrt{\frac{bfz^2}{1+bz^2}}$ relinquo q. Integrale huius $PRdz$ ita capiat
$z = 0$ ipsum evanescat, quo facto ponatur $z = \infty$, denotetqu
provenit, si tantum

$$\int \frac{dz}{(1+bz^2)^{\frac{n+1}{n+2}}}$$

hoc modo integretur, ut fiat $= 0$ posito $z = 0$, et postmodum po
Tum ergo erit integrale ipsius $PRdz$ praescripto modo acceptum
mus Z § 4 functio ipsius f. Aequale id autem erat positum quantit
seriem

$$1 + A\alpha g + AB\alpha\beta g^2 + \text{etc.}$$

multiplicatae, quae series in sequentem est transmutata

$$1 + \frac{f}{(n+1)(n+2)} + \frac{f^2}{(n+1)(n+2)(2n+3)(2n+4)} + \ldots$$

cuius summa est t, vide § 6, ubi f designat ax^{n+2}. Erit ergo $Z =$
est quantitas constans, quia in ea non inest f adeoque nec x. P

$$t = \frac{Z}{H}, \text{ at est } y = \frac{dt}{tdx};$$

ergo pro aequatione proposita

$$ax^n dx = dy + y^2 dx$$

prodibit $y = \frac{dZ}{Zdx}$. Ad illam igitur aequationem construendam
regulam: Integretur haec formula[1])

$$\frac{1}{(n+2)^2} \frac{dz}{(1+bz^2)^{\frac{n+1}{n+2}}} \left(n(n+4) + 2\, c^{\frac{2}{n+2}\sqrt{\frac{bfz^2}{1+bz^2}}} + 2\, c^{\frac{-2}{n+2}\sqrt{\ldots}}\right.$$

---

1) Loco huius formulae substituatur:

$$\frac{dz}{2(1+bz^2)^{\frac{n+1}{n+2}}}\left(c^{\frac{2}{n+2}\sqrt{\frac{bfz^2}{1+bz^2}}} + c^{\frac{-2}{n+2}\sqrt{\frac{bfz^2}{1+bz^2}}}\right)$$

Vide notam 1 p. 31.

od post integrationem debeat fieri $z = \infty$, is loco z substituat $\frac{u}{1-u}$ et
egrationem ponatur $u = 1$, quo facto pro Z idem prodibit valor, qui a
uamvis autem analytica pro Z expressio obtineri non potest, quando for
non est integrabilis, tamen per quadraturas vel rectificationes valor i
construi poterit.

15. Quanquam autem in hac constructione ii casus excluduntur, in qu
ontineatur intra limites -2 et 0, nihilo tamen minus haec solutio pro uni
i est habenda. Nam quia, si aequatio potest resolvi in casu $n = -m$, resol
oque habetur in casu $n = -m-4$, ut constat[1]) ex iis, quae de cas
parabilibus sunt detecta, perspicuum est, si m sit numerus intra limites
2 contentus, fore $-m-4$ intra terminos -2 et -4 comprehensum, adeo
solutione nostra contineri. Quamobrem si occurrat casus, quo n contine
ra 0 et -2, hic statim reducatur ad alium per dictum theorema, qui i
2 et -4 contineatur, huiusque constructio erit in promptu.

16. In formula differentiali § 14 eruta observo, quoties habuerit
iusmodi formam $k + \frac{1}{2}$, ubi k numerum integrum affirmativum den
egram formulam posse integrari [§ 17], et hanc ob rem valorem ipsi
ipsa exhiberi. His igitur in casibus valor ipsius y quoque poterit defini
quatio integrari. Fiet tum autem $n = \frac{-4k}{2k+1}$, quoties ergo n talem hab
rmam, aequatio

$$ax^n dx = dy + y^2 dx$$

egrationem admittet. Deinde quia casus, si $n = \frac{-m}{m+1}$ vel $n = -m$
luci potest ad casum $n = m$, integrabilis etiam erit aequatio, si

$$n = \frac{-4k}{2k+1} \text{ vel } \frac{-4k-4}{2k+1}$$

1) Vide p. 18 huius voluminis et *Institutionum calculi integralis* vol. I, § 436—441 et v
55—966; cf. quoque § 831—841 et § 940—943; LEONHARDI EULERI *Opera omnia*, series I, vol. 11
H

integrabiles vel separabiles, ab aliis iam eruti, ubi videre licet in
mentariis A. 1726.

17. Esse autem aequationem integrabilem, quoties sit

$$\frac{n+1}{n+2} = k + \frac{1}{2},$$

hoc modo ostendo. Pono

$$\frac{bz^2}{1+bz^2} = u^2;$$

erit

$$z = \frac{u}{\sqrt{b(1-u^2)}} \qquad 1 + bz^2 = \frac{1}{1-u^2}$$

ideoque

$$dz = \frac{du}{(1-u^2)^{\frac{3}{2}}\sqrt{b}}.$$

Fiet igitur

$$\frac{dz}{(1+bz^2)^{\frac{n+1}{n+2}}} = \frac{du}{\sqrt{b}}(1-u^2)^{k-1}.$$

Hanc ob rem formula § 14 integranda transformabitur in han

$$\frac{1}{(n+2)^2\sqrt{b}}\Big(n(n+4)du(1-u^2)^{k-1} + 2c^{\frac{2u\sqrt{f}}{n+2}}du(1-u^2)$$

$$2c^{\frac{-2u\sqrt{f}}{n+2}}du(1-u^2)^{k-1}\Big),$$

quae, ut facile perspicitur, re ipsa integrari potest, quoties k
integer affirmativus[2]). Atque hinc non parum praestantiae a
huic meae methodo, quae tam sit facilis et perspicua, ut casus e
reipsa integrationem vel separationem admittunt, uno obtutu

---

1) Loco huius formulae substituatur

$$\frac{1}{2\sqrt{b}}\Big(c^{\frac{2u\sqrt{f}}{n+2}}du(1-u^2)^{k-1} + c^{\frac{-2u\sqrt{f}}{n+2}}du(1-u^2)^{k-1}\Big).$$

Vide notam 1 p. 32.

2) Cf. Commentationem 70 § 14, huius voluminis p. 161.

in hanc

$$\frac{1}{2\sqrt{b}}(c^{-u\sqrt{f}}du + c^{u\sqrt{f}}du),$$

integralis est

$$\frac{1}{2\sqrt{bf}}(c^{u\sqrt{f}} - c^{-u\sqrt{f}}).$$

autem non adiicio, quia posito $z = 0$, seu quod eodem recidit $u = 0$, integrale iam evanescit. Fiat nunc $z = \infty$ seu in nostro casu $u = 1$ et ur ax^{-2} loco f, habebitur

$$Z = \frac{x}{2\sqrt{(ab)}}\left(c^{\frac{\sqrt{a}}{x}} - c^{\frac{-\sqrt{a}}{x}}\right).$$

nvento erit, ut iam est ostensum, $y = \frac{dZ}{Zdx}$. Differentiato igitur Z et differli per Zdx diviso prodibit

$$y = \frac{1}{x} - \frac{\sqrt{a}}{x^2}\left(\frac{c^{\frac{2\sqrt{a}}{x}} + 1}{c^{\frac{2\sqrt{a}}{x}} - 1}\right) \text{ sive } 2\frac{\sqrt{a}}{x} = l\frac{xxy - x - \sqrt{a}}{xxy - x + \sqrt{a}},$$

aequatio est integralis huius differentialis

$$ax^{-4}dx = dy + y^2dx.$$

simili modo pro reliquis casibus, qui separationem admittunt, aequaintegrales inveniuntur.

INFINITIS CURVIS EIUSDEM

Commentatio 44 indicis ENESTROEMIANI
Commentarii academiae scientiarum Petropolitanae 7 (1734/5), 17

1. Curvas eiusdem generis hic voco tales curvas
differunt nisi ratione lineae cuiusdam constantis, quae
assumens eas curvas determinat. Linea haec consta
modulus est vocatus, ab aliis parameter: quia autem
biguitatem creare potest, moduli vocabulum retinel
linea constans et invariabilis, dum una infinitarum
determinatur; varios autem habet valores et ideo va
curvas refertur. Sic si in aequatione $y^2 = ax$ sum
variabilitate ipsius a innumerabiles oriuntur para
positae et communem verticem habentes.

2. Infinitae igitur curvae eiusdem generis o
exprimuntur, quam modulus qui nobis semper littera
Huic enim modulo, si successive alii atque alii valo
continuo alias dabit curvas, quae omnes in una
Aequationem hanc modulum continentem cum HER
bimus; in qua igitur praeter alias constantes et eiu

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1) IAC. HERMANN (1678—1733), *schediasma de traiectoriis dat*
occurrentibus. Acta erud. 1717 p. 348: „per modulum hic intelligo li
demque curvae secandae est constans, sed in diversis curvis eiusdem
G. W. LEIBNIZ, *De linea ex lineis numero infinitis ordinatim ductis*
Acta erud. 1692 p. 168: „parametri seu rectae magnitudine const
aequationis pro ipsa calculum ingredientes, quae per a, b etc. desig

odularem invenire. Nam sit[2]) $z = \int Pdx$, ubi P in a, z et x quomodocun
etur, seu $dz = Pdx$, in qua aequatione a ut constans consideratur; inte
tur aequationem modularem haberi, si integralis aequationis $dz = P$
nuo differentietur, posito etiam a variabili. Sed cum integrationem perfic
on liceat, eiusmodi methodus desideratur, qua differentialis aequatio, q
rodiret, si integralis denuo differentietur posita etiam a variabili, inven
ossit.

4. Ad construendas quidem et cognoscendas curvas aequatio $dz = P$
ufficit. Nam dato ipsi modulo a certo valore construetur aequatio $dz = P$
uo facto habebitur una curvarum infinitarum, eodemque modo aliae re
entur aliis ponendis valoribus loco a. Sed si in his curvis certa puncta debe
ssignari prout problema aliquod postulat, talis aequatio $z = \int Pdx$ n
ufficit sed requiritur aequatio a signis summatoriis libera, in qua si non
gebraica, etiam differentialia ipsius a insint. Ex data igitur aequatio
fferentiali pro unica curva $dz = Pdx$, in qua a ut constans considerat
uaeri oportet aequationem differentialem, in qua et a sit variabilis, haecc
it modularis. Haec vero modularis interdum erit differentialis primi grad
terdum secundi et altioris, interdum etiam penitus non poterit inveniri.

5. Quo igitur methodum tradam, qua ex aequatione different
$z = Pdx$, in qua a est constans, modularis possit inveniri, quae a ut var
lem contineat; pono primo P esse functionem ipsarum a et x tantum,

1) Commoditatis causa et ad posteriorem huius doctrinae usum, EULERUS in hac Commentat
xcepta fortasse § 37) nihil aliud considerat nisi algebraicas ipsorum x, z et a functiones vel integ
nctionum algebraicarum unius variabilis x. Vide § 10, 11, 18, 19, 20, 27, 31. Vide quoque Comm
tionem 45 § 3, 4, 5. Attamen in hac altera Commentatione EULERUS omnis generis functi
cipit. H.

2) Integrale hoc $\int Pdx$ erit, in iis quae sequuntur, functio ipsarum x, z et a ita determinat
anescat posito $x = 0$, vel $x = x_0$, x_0 non pendente ab a. Vide *Institutionum calculi integralis* vo
1017. H.

6. Ad inveniendum autem valorem ipsius Q seq
*Quantitas A ex duabus variabilibus t et u utcunque com
posito t constante hocque differentiale denuo differentietu
variabili, idem resultat ac si inverso ordine A primo dif
stante hocque differentiale denuo differentietur posito t cons*

$$A = \sqrt{(t^2 + nu^2)},$$

differentietur posito t constante, habebitur

$$\frac{nudu}{\sqrt{(t^2 + nu^2)}}.$$

Hoc denuo differentietur posito u constante et prodibit

$$\frac{-ntudtdu}{(t^2 + nu^2)^{\frac{3}{2}}}.$$

Iam ordine inverso differentietur $\sqrt{(t^2 + nu^2)}$ posit
differentiale

$$\frac{tdt}{\sqrt{(t^2 + nu^2)}},$$

quod denuo differentiatum posito t constante dabit

$$\frac{-ntudtdu}{(t^2 + nu^2)^{\frac{3}{2}}},$$

id quod congruit cum prius invento. Atque similis co
aliis exemplis cernetur.

7. Quamvis autem huius theorematis veritatem
ciant, demonstrationem tamen sequentem adiiciam

$+ dt$ loco t et $u + du$ loco u mutetur A in D. Ex his perspicuum es
B scribatur $u + du$ loco u, provenire D; similique modo si in C pon
$+ dt$ loco t proditurum quoque D. His praemissis si differentietur A pos
nstante, prodibit $C - A$, nam posito $u + du$ loco u abit A in C, differen
tem est $C - A$. Si porro in $C - A$ ponatur $t + dt$ loco t prodibit $D -$
are differentiale erit

$$D - B - C + A.$$

verso nunc ordine posito $t + dt$ loco t in A habebitur B, eritque differen
sius A posito tantum t variabili $B - A$. Hoc differentiale posito $u + du$
abit in $D - C$, quare eius differentiale erit

$$D - B - C + A,$$

quod congruit cum differentiali priori operatione invento. Q. E. D.

8. Istud autem theorema hoc modo inservit ad valorem ipsius Q
niendum. Cum P et Q sint functiones ipsarum a et x, sit

$$dP = Adx + Bda \text{ et } dQ = Cdx + Dda,$$

que z cum sit $= \int Pdx$, erit quoque functio ipsarum a et x, positum autem

$$dz = Pdx + Qda.$$

m secundum theorema differentietur z posito x constante eritque diffe
le Qda, hoc porro differentiatum posito a constante dabit $Cdadx$. Al
eratione differentiale ipsius z posito primo a constante est Pdx, huius
fferentiale posito x constante est $Bdadx$. Quare vi theorematis aequalia
bent $Cdadx$ et $Bdadx$, ex quo fit $C = B$. Datur autem B ex P; differen
im ipsius P posito x constante divisum per da dat B. Cum igitur sit

$$dQ = Bdx + Dda,$$

it $Q = \int Bdx$, si in hac integratione a ut constans consideretur[2]).

1) Vide *Institutionum calculi differentialis* vol. I, § 228—240. *Leonhardi Euleri Opera o*
ies I, vol. 10. H.

2) Vide L. Euleri Commentationem 45, huius voluminis p. 57. Cf. *Institutiones c*
egralis vol. I, § 457; vol. II § 1016—1057. *Leonhardi Euleri Opera omnia*, series I, vol. 11
H.

existente

$$dP = Adx + Bda.$$

Si igitur Bdx integrari poterit, desiderata habebitur aeq integrari non potest, aeque inutilis est haec aequatio utraque enim involvit integrationem differentialis, in qu considerari, id quod est contra naturam aequationis mo a aeque variabile esse debet ac x et z.

10. Quando autem Bdx integrationem non aequatio inventa ut inutilis omnino est negligenda. N Bdx pendeat a $\int Pdx$, aequatio modularis poterit exhi

$$\int Bdx = a\int Pdx + K$$

existente K functione ipsarum a et x algebraica, erit ol

$$\int Bdx = az + K$$

et

$$dz = Pdx + azda + Kda,$$

quae aequatio revera erit modularis. Quoties igitur integrari vel ad integrationem ipsius Pdx deduci, a dularis, quae erit differentialis primi gradus. At si Pd quidem opus est, sed $z = \int Pdx$ erit simul aequatio mo

11. Si autem $\int Bdx$ neque algebraice exhiberi potest, dispiciendum est, num $\int Bdx$ ad integrationer quo a non inest, possit reduci. Tale enim integrale, in qu aequationem modularem, cum si libuerit per differentiat eodem iure, si $\int Pdx$ reduci poterit ad aliud integral nequidem hac ipsius Q determinatione opus est, se aequationem modularem, ut si sit

$$\int Pdx = h\int Kdx$$

$$dz = \frac{zdh}{h} + Khdx.$$

Si autem haec omnia nullum inveniant locum, indicio est, aequa-
modularem primi gradus differentialem non dari. Quamobrem in
gradus differentialibus quaeri debebit. Ad hoc differentio denuo
onem

$$dz = Pdx + da\int Bdx.$$

utem

$$dB = Edx + Fda,$$

to erit ipsius $\int Bdx$ differentiale

$$Bdx + da\int Fdx.$$

ntiatione igitur peracta et loco $\int Bdx$ eius valore ex eadem aequatione
$\frac{dz}{da} - \frac{Pdx}{da}$ posito, habebitur

$$dz = Pddx + dPdx + \frac{dzdda}{da} - \frac{Pdxdda}{da} + Bdadx + da^2\int Fdx.$$

itur

$$\int Fdx = \frac{ddz}{da^2} - \frac{dzdda}{da^3} - \frac{Pddx}{da^2} - \frac{dPdx}{da^2} + \frac{Pdxdda}{da^3} - \frac{Bdx}{da}.$$

utem sit $\int Bdx = \frac{dz}{da} - \frac{Pdx}{da}$ et $\int Pdx = z$, si $\int Fdx$ reduci poterit ad
dia $\int Bdx$ et $\int Pdx$ vel si reipsa poterit integrari, habebitur aequatio
aris, quae erit differentialis secundi gradus. Ut si fuerit

$$\int Fdx = \alpha\int Bdx + \beta\int Pdx + K,$$

α et β utcumque per a et constantes, et K per a et x et constantes, erit
io modularis haec

$$\frac{daddz - dzdda - Pdaddx + Pdxdda - dPdadx}{da^3} - \frac{Bdx}{da} = \frac{\alpha dz - \alpha Pdx}{da} + \beta z + K.$$

t F ex dato P facile reperiuntur.

13. Si $\int F dx$, quod autem rarissime evenit, vel non amp
tineat a, vel ad aliud possit reduci, in quo a non insit, aequatio
rentialis secundi gradus pro legitima modulari poterit haber
omnia nondum succedant, adhuc differentiatio est instituenda,
rentiale ipsius $\int F dx$ erit

$$F dx + da \int H dx$$

posito

$$dF = G dx + H da.$$

Quo facto videndum est, vel an $\int H dx$ re ipsa possit exhiberi, ve
praecedentibus $\int F dx$, $\int B dx$ et $\int P dx$, vel an possit ex sign
a eliminari. Quorum si quod obtigerit, habebitur aequatio mod
tialis tertii gradus; sin vero nullum locum habuerit, continuan
tiatio simili modo, donec signa summatoria potuerint eliminari

14. His generalibus praemissis ad specialia accedo, casn
quibus functio P quodammodo determinatur. Sit igitur P fu
tantum, a prorsus non involvens, quam littera X designabo, erit e
quae quidem aequatio quia non continet a, ad unicam videtur cur
neque ad modularem praebendam apta esse. Sed cum in inte
stantem addere liceat, poterit esse

$$z = \int X dx + na$$

seu differentiando

$$dz = X dx + n da,$$

quae est vera aequatio modularis. Eadem aequatio prodiis
regulam X differentiassem posito x constante, unde prodit $B =$
constanti, orta igitur esset aequatio modularis

$$dz = X dx + n da,$$

cuius loco potius integralis

$$z = \int X dx + na$$

usurpatur.

15. Sit nunc $P = AX$, existente A functione ipsius a e
tantum. Cum igitur sit $z = \int P dx$, erit $z = \int AX dx$ seu quia in
a ut constans debet considerari, $z = A \int X dx$. Quae aequatio s
rentialis

n, poni potest ipse modulus a, nam loco moduli eius functio quaecunque iure pro modulo haberi potest.

6. Sit $P = A + X$ litteris A et X eosdem ut ante retinentibus valores. Ergo

$$dz = Adx + Xdx$$

$z = Ax + \int Xdx$, quae aequatio iam est modularis, quia modulus A st in signo summatorio involutus. Si quem autem $\int Xdx$ offendat, ntialem aequationem

$$dz = Adx + xdA + Xdx$$

odulari habere potest.

7. Simili ratione modularem aequationem invenire licet, si fuerit

$$P = AX + BY + CZ + \text{etc.},$$

B, C etc. sunt functiones quaecunque ipsius moduli a et X, Y, Z etc. ones quaecunque ipsius x et constantium excepta a. Namque ob

$$dz = AXdx + BYdx + CZdx + \text{etc.}$$

$$z = A\int Xdx + B\int Ydx + C\int Zdx + \text{etc.},$$

imul est modularis, cum modulus a nusquam post signum summatorium atur.

8. Sit $P = (A + X)^n$ seu $z = \int dx\,(A + X)^n$. Differentiale ipsius P x constante est $n\,(A + X)^{n-1}dA$, id quod per da divisum dat superiorem m B (vide § 8). Erit igitur

$$dz = (A + X)^n dx + ndA\int (A + X)^{n-1}dx$$

$$\int dx\,(A + X)^{n-1} = \frac{dz - (A + X)^n dx}{ndA}.$$

igitur sit

$$\int dx\,(A + X)^n = z,$$

etiam exprimi poterit, habebitur quod quaeritur. Si neutru
differentiatio est instituenda. Est autem differentiale ipsi

$$dx(A + X)^{n-1} + (n-1)\,dA\int(A+X)^{n-2}dx = \text{Diff.}\ \frac{d}{}$$

Erit itaque

$$\int dx\,(A+X)^{n-2} = \frac{1}{(n-1)dA}\ \text{Diff.}\ \frac{dz-(A+X)^n dx}{ndA}$$

Quare videndum est, an $\int dx\,(A+X)^{n-2}$ possit vel inte
integralia reduci.

19. Si n fuerit numerus integer affirmativus, aequ
algebraica. Nam $(A+X)^n$ potest in terminos numero fin
quisque in dx ductus integrari potest, ita ut modulus a in si
non ingrediatur. Erit autem aequatio modularis haec

$$z = A^n x + \frac{n}{1}A^{n-1}\int X dx + \frac{n(n-1)}{1\cdot 2}A^{n-2}\int X^2$$

Examinandum igitur restat, quibus casibus, si n non fu
affirmativus, supra memoratae conditiones locum habean

20. Sit primo $X = bx^m$, ubi b etiam ab a pond
$z = \int(A+bx^m)^n dx$. Haec formula primo ipsa est in
designante i numerum quemcunque affirmativum integ
$m = \frac{-1}{n+i}$. His igitur casibus aequatio modularis fit algebr
ubi b ab a non pendere potest, illa quidem aequatio i
mittit sed sequens

$$dz = \left(A + bx^{\frac{-1}{n}}\right)^n dx + ndA\int dx\left(A + b\right.$$

evadit integrabilis fitque aequatio modularis differential

1) Si litterae i valores negativi attribuuntur, integrale terminis
Aequatio modularis dicitur iis tantum casibus, quibus integrale algebrai

$$z = \int x^m dx\,(A + bx^k)^n.$$

et enim

$$dz = x^m dx\,(A + bx^k)^n + ndA\int x^m dx\,(A + bx^k)^{n-1}.$$

d est

$$\int x^m dx\,(A + bx^k)^n = \frac{x^{m+1}(A + bx^k)^n}{m + nk + 1} + \frac{nkA}{m + nk + 1}\int x^m dx\,(A + bx^k)^{n-}$$

1

$$\int x^m dx\,(A + bx^k)^{n-1} = \frac{(m + nk + 1)z}{nkA} - \frac{x^{m+1}(A + bx^k)^n}{nkA}.$$

nsequenter habebitur aequatio modularis haec

$$Akdz = (A + bx^k)^n\,(Akx^m dx - x^{m+1}dA) + (m + nk + 1)\,zdA.$$

nili modo modularis esset inventa, si fuisset

$$z = B\int x^m dx\,(A + bx^k)^n,$$

a enim non prodiisset differentia, nisi quod loco z scribi debuisset $\frac{z}{B}$ et

$\frac{Bdz - zdB}{B^2}$, si quidem B ab a etiam pendeat.

22. Missis autem huiusmodi litterae P determinationibus, quippe q
nus late patent, ad alias accedo, quae multo saepius usui esse possunt. C
entur hae determinationes ea functionis cuiusdam propositae propriet
a functio eundem ubique tenet dimensionum quantitatum variabil
merum. Tales enim functiones peculiari modo differentiationem admitt
sit u functio nullius dimensionis ipsarum a et x, cuiusmodi sunt $\frac{a}{x}$, $\frac{\sqrt{(a^2 -}}{a}$
aeque similes, in quibus ipsarum a et x dimensionum numerus in de
natore aequalis est numero dimensionum numeratoris. Det autem
nctio u differentiata $Rdx + Sda$; dico fore

$$Rx + Sa = 0.$$

m si in functione u ponatur $x = ay$, omnia a sese destruent et in ea pra
et constantes nulla alia littera remanebit. Hanc ob rem in differentiali
nc substitutionem aliud differentiale praeter dy non reperietur. Cum au
$x = ay$, erit $dx = ady + yda$, ideoque

$$du = Rady + Ryda + Sda.$$

Debebit ergo esse

$$Ry + S = 0 \text{ seu } Rx + Sa = 0.$$

23. Sin vero fuerit u functio m dimensionum ipsarum a e

$$du = Rdx + Sda,$$

erit $\frac{u}{x^m}$ functio ipsarum a et x nullius dimensionis. Differentiet prodibit

$$\frac{xdu - mudx}{x^{m+1}} \text{ seu } \frac{Rxdx - mudx + Sxda}{x^{m+1}}.$$

Quod cum sit differentiale functionis nullius dimensionis, erit

$$Rx^2 - mux + Sax = 0$$

seu

$$Rx + Sa = mu.$$

Quare si fuerit u functio m dimensionum ipsarum a et x, atq

$$du = Rdx + Sda,$$

erit

$$Rx + Sa = mu$$

ideoque

$$du = Rdx + \frac{da}{a}(mu - Rx)$$

seu

$$adu = Radx - Rxda + muda.$$

24. His praemissis in $dz = Pdx$ seu $z = \int Pdx$ sit P fun sionum ipsarum a et x, erit igitur z talis functio dimensionum si ponatur $dz = Pdx + Qda$, erit

$$Px + Qa = (n + 1)z.$$

Ex quo valor ipsius Q substitutus dabit aequationem modular

$$dz = Pdx + \frac{da}{a}((n + 1)z - Px)$$

seu

$= \int Bdx$, erit hoc casu

$$(n+1)\int Pdx = a\int Bdx + Px.$$

Ex quo perspicitur hoc casu integrale $\int Bdx$ semper reduci ad $\int Pdx$.

25. Eadem aequatio modularis proveniet ex consideratione sol

Posito enim $dP = Adx + Bda$, erit

$$nP = Ax + Ba.$$

um autem sit

$$dz = Pdx + da\int Bdx,$$

rit

$$dz = Pdx + \frac{da}{a}\int(nPdx - Axdx),$$

n qua integratione a constans habetur. Erit igitur $\int nPdx = nz$, et

$$\int Axdx = Px - \int Pdx,$$

b $\int Adx = P$. Habebitur itaque

$$dz = Pdx + \frac{da}{a}((n+1)z - Px),$$

l quod prorsus congruit cum praecedentibus.

26. Retinente P suum valorem n dimensionum sit $z = \int APXd$

l sit functio ipsius a et X ipsius x tantum. Erit igitur $\frac{z}{A} = \int PXdx$.

$$dP = Adx + Bda,$$

n quo littera A cum altera, quae est functio ipsius a tantum, non est c
enda), erit

$$nP = Ax + Ba.$$

psius PX differentiale igitur posito x constante erit $BXda$. Conseq
abebitur

$$d\cdot\frac{z}{A} = PXdx + da\int BXdx = PXdx + \frac{da}{a}\int(nPXdx - AXxdx).$$

Quare fiet

$$d \cdot \frac{z}{A} = PXdx - \frac{PXxda}{a} + \frac{(n+1)zda}{Aa} + \frac{d}{}$$

Nisi igitur $\int PxdX$ reduci poterit ad $\int PXdx$ vel pror modularis differentialis primi gradus dari nequit.

27. At si fuerit $z = R\int Pdx$, existente R functione ex a, x et etiam ex z constante, at P functione ipsarum quia est $\frac{z}{R} = \int Pdx$, erit

$$d \cdot \frac{z}{R} = Pdx + \frac{da}{a}\left(\frac{(n+1)z}{R} - Px\right) = \frac{Ra}{}$$

seu

$$Radz - zadR - (n+1)Rzda = PR^2adx -$$

In universum autem teneatur, quoties $z = \int Pdx$ ad ae reduci possit, toties etiam $z = R\int Pdx$ ad aequation posse. Nullum aliud enim discrimen aderit, nisi quod casu debeat esse $\frac{z}{R}$. Quare si R fuerit vel quantitas alg cendens, ut eius differentiale posito etiam a variabili exhiberi, aequatio modularis per praecepta data repe posterum tales casus, etiamsi latius pateant, praeterm

28. Ponamus esse

$$z = \int(P+Q)dx \text{ seu } z = \int Pdx + \int Q$$

et P esse functionem ipsarum a et x dimensionum n - earundem a et x dimensionum $m-1$. Cum igitur diffe

$$\frac{P(adx - xda)}{a} + \frac{da}{a}\int nPdx$$

et differentiale ipsius $\int Qdx$ sit

$$\frac{Q(adx - xda)}{a} + \frac{da}{a}\int mQdx,$$

$$\frac{adz - (P + Q)(adx - xda)}{da} = u,$$

$$u = n\int Pdx + m\int Qdx.$$

porro differentietur erit

$$du = \frac{(nP + mQ)(adx - xda)}{a} + \frac{da}{a}(n^2\int Pdx + m^2\int Qdx).$$

gitur

$$\frac{adu - (nP + mQ)(adx - xda)}{da} = t$$

$$t = n^2\int Pdx + m^2\int Qdx.$$

is nunc ex his tribus aequationibus ipsarum z, u et t integralibus $\int Qdx$ prodibit haec aequatio

$$mnz - (m + n)u + t = 0.$$

quatio, si loco u et t valores assumti substituantur, erit modularis .

Simili modo, si fuerit

$$z = \int (P + Q + R)dx$$

ctio $n - 1$, Q functio $m - 1$ et R functio $k - 1$ dimensionum ipsarum ponatur

$$u = \frac{adz - (P + Q + R)(adx - xda)}{da}$$

$$t = \frac{adu - (nP + mQ + kR)(adx - xda)}{da}$$

$$s = \frac{adt - (n^2P + m^2Q + k^2R)(adx - xda)}{da}.$$

to erit aequatio modularis haec:

$$kmnz - (km + kn + mn)u + (k + m + n)t - s = 0.$$

30. Sit porro

$$z = \int (P+Q)^k dx,$$

ubi P sit functio n dimensionum, Q vero functio m dimensionum ipsa
Quando igitur est

$$dP = Adx + Bda \text{ et } dQ = Cdx + Dda,$$

erit

$$nP = Ax + Ba \text{ et } mQ = Cx + Da.$$

Differentiale autem ipsius $(P+Q)^k$ posito x constante divisum $k(B+D)(P+Q)^{k-1}$. Hanc ob rem erit

$$dz = (P+Q)^k dx + \frac{kda}{a}\int (P+Q)^{k-1}(Ba+Da)dx.$$

Cum autem sit

$$Ba = nP - Ax \text{ et } Da = mQ - Cx$$

$$\text{et } Adx = dP \text{ et } Cdx = dQ,$$

ob a in hac integratione constans, erit

$$dz = (P+Q)^k dx + \frac{kda}{a}\int (P+Q)^{k-1}(nPdx + mQdx - xdP -$$

seu

$$dz = \frac{(P+Q)^k(adx - xda)}{a} + \frac{da}{a}\int (P+Q)^{k-1}((nk+1)Pdx + (mk -$$

Ponatur

$$\frac{adz - (P+Q)^k(adx - xda) - zda}{kda} = u$$

erit

$$u = \int (nPdx + mQdx)(P+Q)^{k-1}.$$

Quare si integrale $\int (nPdx + mQdx)(P+Q)^{k-1}$ pendet ab integrali $\int($
habebitur aequatio modularis differentialis gradus primi; sin minu
tiatio est continuanda. Fit autem

$$du = (nPdx + mQdx)(P+Q)^{k-1} + \frac{uda}{a} - \frac{da}{a}(nP + mQ)(P +$$

$$+ \frac{da}{a}\int (kn^2P^2dx + (2kmn + n^2 - 2mn + m^2)PQdx + km^2Q^2dx)(P$$

$$= \int (kn^2P^2dx + (2kmn + n^2 - 2mn + m^2)\, PQdx + km^2Q^2dx)\, (P + Q$$

31. Cum igitur habeantur tria integralia, videndum est, num ea a s
em pendeant, hoc enim si fuerit, habebitur aequatio algebraica inter t, u
e dabit loco t et u substitutis assumtis valoribus aequationem modula
erentialem secundi gradus. Quo autem facilius in casibus particularibus
ci possit, an pendeant a se invicem, ad alias formas eas reduci conv
n igitur sit $z = \int (P + Q)^k dx$, erit

$$u = mz + (n - m) \int (P + Q)^{k-1} Pdx$$

$$(2km + n - m)\, u - (km^2 - m^2 + mn)\, z + (n - m)^2 (k - 1) \int (P + Q)^{k-2} P$$

aerendum itaque est an

$$\int (P + Q)^{k-2} P^2 dx$$

uci possit ad haec $\int (P + Q)^{k-1} Pdx$ et $\int (P + Q)^k dx$ vel an sit

$$\int (P + Q)^{k-2} P^2 dx = \alpha \int (P + Q)^{k-1} Pdx + \beta \int (P + Q)^k dx + V$$

ignante V quantitatem algebraicam quamcunque per a et x datam, et α
t coefficientes ex constantissimis et a compositi.

32. Fiat igitur $V = T(P + Q)^{k-1}$, huius differentiale posito a const

$$dT(P + Q)^{k-1} + (k - 1)\,(TdP + TdQ)\,(P + Q)^{k-2}.$$

dibit ergo sequens aequatio

$$^2dx = \alpha P^2dx + \alpha PQdx + \beta P^2dx + 2\beta PQdx + \beta Q^2dx + PdT + Q$$
$$+ (k - 1)\, TdP + (k - 1)\, TdQ,$$

e per dx dividi poterit. At T ita debet accipi, ut termini respondentes
truant, sumtis ad hoc idoneis pro α et β valoribus.

33. At si per $\int Pdx$ non absolute determinetur z sed quantitas $\int Qdz$,
tcunque per a et z, atque P per a et x, habebitur ista aequatio $Qdz = $

ubi P sit functio n dimensionum, Q vero functio m dimensio
Quando igitur est

$$dP = Adx + Bda \text{ et } dQ = Cdx + Dda,$$

erit

$$nP = Ax + Ba \text{ et } mQ = Cx + Da.$$

Differentiale autem ipsius $(P + Q)^k$ posito x constante d
$k(B + D)(P + Q)^{k-1}$. Hanc ob rem erit

$$dz = (P + Q)^k dx + \frac{kda}{a}\int(P + Q)^{k-1}(Ba +$$

Cum autem sit

$$Ba = nP - Ax \text{ et } Da = mQ - Cx$$

$$\text{et } Adx = dP \text{ et } Cdx = dQ,$$

ob a in hac integratione constans, erit

$$dz = (P + Q)^k dx + \frac{kda}{a}\int(P + Q)^{k-1}(nPdx + mQdx$$

seu

$$dz = \frac{(P + Q)^k(adx - xda)}{a} + \frac{da}{a}\int(P + Q)^{k-1}((nk + 1)Pdx$$

Ponatur

$$\frac{adz - (P + Q)^k(adx - xda) - zda}{kda} = u$$

erit

$$u = \int(nPdx + mQdx)(P + Q)^{k-1}.$$

Quare si integrale $\int(nPdx + mQdx)(P + Q)^{k-1}$ pendet ab in
habebitur aequatio modularis differentialis gradus primi;
tiatio est continuanda. Fit autem

$$du = (nPdx + mQdx)(P + Q)^{k-1} + \frac{uda}{a} - \frac{da}{a}(nP +$$

$$+ \frac{da}{a}\int(kn^2P^2dx + (2kmn + n^2 - 2mn + m^2)PQdx + kn$$

$$\frac{\quad}{da}$$

erit

$$t = \int (kn^2P^2dx + (2kmn + n^2 - 2mn + m^2)\,PQdx + km^2Q^2dx)\,(P$$

31. Cum igitur habeantur tria integralia, videndum est, num ea
vicem pendeant, hoc enim si fuerit, habebitur aequatio algebraica inter
quae dabit loco t et u substitutis assumtis valoribus aequationem mo
differentialem secundi gradus. Quo autem facilius in casibus particulari
spici possit, an pendeant a se invicem, ad alias formas eas reduci c
Cum igitur sit $z = \int(P + Q)^k dx$, erit

$$u = mz + (n - m)\int(P + Q)^{k-1}Pdx$$

et

$$t = (2km + n - m)u - (km^2 - m^2 + mn)z + (n - m)^2(k - 1)\int(P + Q)^k$$

Quaerendum itaque est an

$$\int(P + Q)^{k-2}P^2dx$$

reduci possit ad haec $\int(P + Q)^{k-1}Pdx$ et $\int(P + Q)^k dx$ vel an sit

$$\int(P + Q)^{k-2}P^2dx = \alpha\int(P + Q)^{k-1}Pdx + \beta\int(P + Q)^k dx + V$$

designante V quantitatem algebraicam quamcunque per a et x datam, e
sunt coefficientes ex constantissimis et a compositi.

32. Fiat igitur $V = T(P + Q)^{k-1}$, huius differentiale posito a co
sit

$$dT(P + Q)^{k-1} + (k - 1)(TdP + TdQ)(P + Q)^{k-2}.$$

Prodibit ergo sequens aequatio

$$P^2dx = \alpha P^2dx + \alpha PQdx + \beta P^2dx + 2\beta PQdx + \beta Q^2dx + PdT +$$
$$+ (k - 1)\,TdP + (k - 1)\,TdQ,$$

quae per dx dividi poterit. At T ita debet accipi, ut termini responden
destruant, sumtis ad hoc idoneis pro α et β valoribus.

33. At si per $\int Pdx$ non absolute determinetur z sed quantitas $\int Qd$
Q utcunque per a et z, atque P per a et x, habebitur ista aequatio Qdz

membrum ponendo etiam a variabili ope

$$dP = Adx + Bda \text{ et } dQ = Cdz + Dda.$$

Erit ergo

$$Qdz + da\int Ddz = Pdx + da\int Bdx$$

seu

$$Qdz = Pdx + da\left(\int Bdx - \int Ddz\right).$$

Quae aequatio, si $\int Bdx$ et $\int Ddz$ poterunt eliminari, dab
quaesitam.

34. Sit P functio $m-1$ dimensionum ipsarum a et x, et Q
dimensionum ipsarum a et z. His positis erit

$$\text{Diff.} \int Pdx = \frac{mda\int Pdx + P(adx - xda)}{a}$$

$$\text{et Diff.} \int Qdz = \frac{nda\int Qdz + Q(adz - zda)}{a}.$$

Ex quo eruitur ista aequatio

$$(m-n)\int Pdx = \frac{Q(adz - zda)}{da} - \frac{P(adx - xda)}{da}$$

ob

$$\int Pdx = \int Qdz.$$

Quare si fuerit $m = n$, erit

$$Qadz - Qzda = Padx - Pxda,$$

quae est aequatio modularis seu

$$\frac{da}{a} = \frac{Qdz - Pdx}{Qz - Px}.$$

35. Sin vero m et n non sint aequales, aequatio modularis e
secundi gradus. Nam cum sit

$$(m-n)\int Pdx = \frac{Q(adz - zda) - P(adx - xda)}{da}$$

---

1) In editione principe numeri 180—189 falso iterantur.

$$\frac{Q(adz-zda)-P(adx-xda)}{da}=\frac{m(m-n)da\int Pdx}{a}+\frac{(m-n)P(adx-xda)}{a}$$
$$=\frac{mQ(adz-zda)-nP(adx-xda)}{a}.$$

o aequatio est modularis quaesita.

36. Si in aequatione proposita $dz+Pdx=0$ indeterminatae non fuerint invicem separatae, ita ut P sit functio involvens x et z et a, debebit per titatem quandam R multiplicari, quo formula $Rdz+PRdx$ ut differen- integralis cuiusdam S possit considerari. Erit itaque $dS=Rdz+PRdx=0$ que $S=$ Const. Sed ad quantitatem R inveniendam sit

$$dP=Adx+Bdz \text{ et } dR=Ddx+Edz,$$

a tantisper pro constante habemus. His positis erit

$$d\cdot PR=(DP+AR)dx+(EP+BR)dz,$$

irca debet esse

$$D=EP+BR.$$

),

$$D=\frac{dR-Edz}{dx}$$

$$Edz+EPdx+BRdx=dR.$$

vero sit $dz+Pdx=0$, habebitur

$$dR=BRdx, \text{ et } lR=\int Bdx.$$

itum vero est B ex dato P, et quia B et z et x involvit, Bdx integrari debet aequationis $dz+Pdx=0$, si quidem fieri potest. Sit itaque $\int Bdx=K$, ne $R=e^K$ posito $le=1$.

7. Cum igitur sit

$$dS=e^K dz+e^K Pdx=0,$$

equationem modularem inveniendam sit

$$dK=Fdx+Gdz+Hda,$$

ue

$$de^K=e^K(Fdx+Gdz+Hda).$$

$$e^K dz + e^K Pdx + da\int e^K Hdz = 0,$$

seu diviso per e^K haec

$$dz + Pdx + e^{-K} da\int e^K Hdz = 0.$$

Alia aequatio modularis invenitur posito

$$dP = Adx + Bdz + Cda,$$

erit enim ipsius $e^K P$ differentiale posito x et z constante hoc $e^K (Cda$
Integretur $e^K dx(C + PH)$ posito tantum x variabili, quo facto erit
modularis

$$dz + Pdx + e^{-K} da\int e^K dx(C + PH) = 0.$$

Sed huiusmodi aequationes modulares, nisi R possit sine aequatione
$dz + Pdx = 0$ determinari, nullius fere sunt usus.

38. Consideremus igitur casus particulares, sitque in a
$dz + Pdx = 0$ P functio nullius dimensionis ipsarum x et z, non c
constantibus et modulo a. Formula vero $dz + Pdx$ integrabilis sempe
si dividatur per $z + Px$, quamobrem erit

$$S = \int \frac{dz + Pdx}{z + Px} = \text{Const.}$$

Fit autem

$$\int \frac{dz + Pdx}{z + Px} = l(z + Px) - \int \frac{xdP}{z + Px}.$$

Deinde posito $z = tx$, fiet P functio ipsius t tantum quae sit T. Qu

$$S = l\,(z + Px) - \int \frac{dT}{t + T},$$

quod per quadraturas potest exhiberi.

39. Ad aequationem modularem igitur inveniendam nil aliud
dum, nisi ut $\int \frac{dz + Pdx}{z + Px}$ differentietur posito quoque modulo a varia
tur igitur

$$dP = Adx + Bdz + Cda,$$

osito tantum a variabili, erit eius differentiale $\frac{Czda}{(z+Px)^2}$. Deinde integr $\frac{Czdx}{(z+Px)^2}$ tantum x pro variabili habita, quo facto erit aequatio modu
uaosita

$$dz + Pdx + (z+Px)da\int\frac{Czdx}{(z+Px)^2} = 0.$$

imili modo ex coefficiente ipsius dz qui est $\frac{1}{z+Px}$ prodit haec aequ
odularis

$$dz + Pdx - (z+Px)\,da\int\frac{Cxdz}{(z+Px)^2} = 0,$$

qua integratione z tantum pro variabili habetur. Sive etiam haec

$$dz + Pdx = (z+Px)da\int\frac{Ddt}{(t+T)^2},$$

qua D et T per solum t et a dantur.

40. Praetermittere hic non possum, quin generalem aequationum ho
eorum, uti a CEL. IOH. BERNOULLI[1]) vocantur, quae omnes hac aequat
$z + Pdx = 0$ continentur, resolutionem adiiciam. Namque reperitur ex

$$l(z+Px) = \int\frac{dT}{t+T} = l(t+T) - \int\frac{dt}{t+T},$$

i $t = \frac{z}{x}$ et $T = P$. Prodibit igitur

$$lx + \int\frac{dt}{t+T} = 0$$

u adiecta constante

$$l\frac{c}{x} = \int\frac{dt}{t+T}.$$

t si proposita sit aequatio

$$nxdz + dx\sqrt{(x^2+z^2)} = 0,$$

1) IOH. BERNOULLI, *De integrationibus aequationum differentialium sine praevia indetern
um separatione.* Comment. acad. sc. Petrop. I, 1726, p. 175. *Opera omnia*, t. 3, p. 115. H.

$$l\frac{c}{x} = \int \frac{ndt}{nt + \sqrt{(1 + tt)}};$$

fiat

$$\sqrt{(1 + tt)} = t + s,$$

erit

$$t = \frac{1 - ss}{2s} \text{ et } dt = \frac{-ds(1 + ss)}{2ss}.$$

Quare erit

$$l\frac{c}{x} = \int \frac{-nds(1 + ss)}{(n + 1)s - (n - 1)s^3} = \frac{-n}{n + 1} ls + \frac{n^2}{n^2 - 1} l[(n - 1)s^2 - n$$

41. Quo tamen usus calculi § 36 in casu speciali appareat, si proposita

$$dz + pzdx - qdx = 0,$$

in qua p et q utcunque in a et x dantur. Quae aequatio cum il $dz + Pdx = 0$ collata dat $P = pz - q$, ex quo fiet $B = p$ et seu $R = e^{\int pdx}$. Cum igitur $\int pdx$ per quadraturas possit assignari, valor ipsius R, ideoque aequatio proposita per $e^{\int pdx}$ multiplicat grabilis; erit igitur

$$e^{\int pdx}dz + e^{\int pdx}pzdx - e^{\int pdx}qdx = 0$$

huiusque integralis

$$e^{\int pdx}z = \int e^{\int pdx}qdx \text{ seu } z = e^{-\int pdx}\int e^{\int pdx}qdx.$$

Differentiari itaque debet $e^{-\int pdx}\int e^{\int pdx}qdx$ positis et a et x varia differentiale ipsi dz aequale poni, quo facto habebitur aequatio Positis igitur

$$dp = fdx + gda \text{ et } dq = hdx + ida$$

prodibit ista aequatio modularis

$$dz = -e^{-\int pdx}(pdx + da\int gdx)\int e^{\int pdx}qdx + qdx + e^{-\int pdx}da\int$$
$$(idx + qdx\int gdx),$$

seu posito brevitatis gratia $\int e^{\int pdx}qdx = T$ erit

$$dz = -e^{-\int pdx}Tpdx + qdx + e^{-\int pdx}da\int e^{\int pdx}idx - e^{-\int pdx}da$$

Ex qua operatione intelligi potest ad aequationem modularem in id maxime esse efficiendum, ut in aequatione proposita indeterm invicem separentur.

ADDITAMENTUM AD DISSERTATIONEM DE INFINITIS CURVIS EIUSDEM GENERIS

Commentatio 46 indicis ENESTROEMIANI

Commentarii academiae scientiarum Petropolitanae 7 (1734/5), 1740, p. 184—200

1. In superiore dissertatione[1]), in qua methodum tradidi aequati
o infinitis curvis eiusdem generis inveniendi, ipsius Q valorem in aequa

$$dz = Pdx + Qda$$

terminare docui ex data aequatione $z = \int Pdx$. Namque si P ex x et
nstantibus utcunque fuerit compositum, manifestum est, si $\int Pdx$ differ
r posito non solum x sed etiam a variabili, prodituram esse huius fo
quationem $dz = Pdx + Qda$, in qua valor ipsius Q necessario a quan
quae est cognita, pendebit. Demonstravi scilicet, si differentiale ips
sito x constante fuerit Bda, fore ipsius Q differentiale posito a cons
lx, ex quo pendentia ipsius Q a P satis perspicitur.

2. Cum autem inventus fuerit valor ipsius Q, aequatio

$$dz = Pdx + Qda$$

primet naturam infinitarum curvarum ordinatim datarum, quarum sin
rsim continentur aequatione $dz = Pdx$, a se invicem vero differunt
ate parametri seu moduli a. Et hanc ob rem aequationem $dz = Pdx +$
qua modulus a tanquam quantitas variabilis inest, cum CEL. HERM
quationem modularem vocavi.

1) Vide p. 36. Vide quoque notam p. 39 adiectam. 1

adsunt differentialia, modulus a aeque variabilis ac x et z p
Sin autem Pdx integrari nequit, aequatio etiam modularis n
exceptis casibus, quibus est

$$P = AX + BY + CZ + \text{ etc.},$$

existentibus A, B, C etc. functionibus ipsius a et constanti
etc. functionibus ipsius x et constantium tantum, modul
grediente. Etiamsi enim ipsa aequatio $dz = Pdx$ sit diff
aequatio modularis

$$z = A\int Xdx + B\int Ydx + C\int Zdx + \text{ etc.}$$

instar algebraicae est consideranda.

4. Nisi autem P talem habuerit valorem, aequatio
differentialis gradus primi vel altioris gradus. Differentia
gradus erit, si Q vel erit quantitas algebraica, vel integrale i
hoc enim casu z loco $\int Pdx$ substitutum tollet quoque sign
ita ut aequatio modularis differentialis pura sit proditura.

5. Deprehendi vero in superiore dissertatione Q t
habere valorem, quoties P talis fuerit ipsarum a et x fu
dimensionum, quas a et x constituunt, sit ubique idem atqu
Px vel Pa fuerit functio ipsarum a et x nullius dimensio
observavi [§ 24], quoties in P litterae a et x eundem tantum
dimensionum numerum, toties Q ab integratione ipsius Pdx
cum tam eximia consequantur subsidia ad aequationes modu
maxime iuvabit investigare, num forte aliae dentur hui
ipsius P, quae iisdem praerogativis gaudeant. Has igitur a
constitui, quo simul methodus tales functiones inveniendi a

6. Si P est functio ipsarum a et x dimensionum
ipsarum a et x nullius dimensionis, ostendi fore

$$Px + Qa = 0 \text{ seu } Q = -\frac{Px}{a}.$$

x. Ac si $Q = -\frac{Px}{a}$, erit

$$dz = Pdx - \frac{Pxda}{a}.$$

...amobrem P talis esse debebit functio ipsarum a et x, ut $dx - \frac{xda}{a}$ per ...tiplicatum evadat integrabile. Hic autem per integrabile non s... ...lligo, quod integratione ad quantitatem algebraicam, sed etiam ... quadraturam quamcunque reducitur. Si igitur generaliter invener... ...ntitatem, in quam $dx - \frac{xda}{a}$ ductum fit integrabile, ea erit quaesitus ...us P, eius proprietatis, ut sit $Q = -\frac{Px}{a}$.

7. Fit autem $dx - \frac{xda}{a}$ integrabile, si multiplicatur per $\frac{1}{a}$, integrale ... $\frac{x}{a} + c$, designante c quantitatem constantem quamcunque ab a ...dentem. Quocirca, si $f\left(\frac{x}{a} + c\right)$ denotet functionem quamcunque[1] i... ... c, fiet quoque $dx - \frac{xda}{a}$ integrabile, si multiplicetur per $\frac{1}{a} f\left(\frac{x}{a} + c\right)$. ...r cum sit maxime generalis, erit

$$P = \frac{1}{a} f\left(\frac{x}{a} + c\right) \text{ et } Q = -\frac{Px}{a}.$$

... vero $f\left(\frac{x}{a} + c\right)$ functio quaecunque ipsarum a et x nullius dimensi... ...amobrem quoties Pa fuerit functio nullius dimensionis ipsarum a ... ...es erit $Q = -\frac{Px}{a}$, ideoque aequatio modularis

$$dz = Pdx - \frac{Pxda}{a}.$$

---

1) Hic Eulerus per characteres $f\left(\frac{x}{a} + c\right)$, $f\frac{x}{a}$ functiones ipsius $\frac{x}{a} + c$ vel $\frac{x}{a}$ denotat... ...a per characteres $f.y$, $\varphi:(x+ny)$ functiones ipsorum y, $x + ny$ denotat. Vide Co... ...nem 285 huius voluminis, § 24, 28, 38, 41. H...

$$dz = Pdx + Ada - \frac{Pxda}{a}$$

seu

$$dz - Ada = Pdx - \frac{Pxda}{a}.$$

In qua aequatione cum $dz - Ada$ sit integrabile, debebit $Pdx -$
esse integrabile. Hoc autem per praecedentem operatione
$P = \frac{1}{a} f\left(\frac{x}{a} + c\right)$. Tum igitur erit

$$Q = A - \frac{x}{a^2} f\left(\frac{x}{a} + c\right).$$

Simili ratione intelligitur, si fuerit

$$P = X + \frac{1}{a} f\left(\frac{x}{a} + c\right),$$

denotante X functionem ipsius x tantum, fore

$$Q = A - \frac{x}{a^2} f\left(\frac{x}{a} + c\right),$$

ubi ut ante $f\left(\frac{x}{a} + c\right)$ exprimit functionem quamcunque ipsarum
dimensionis.

9. Sit $Q = -\frac{nPx}{a}$, ubi n indicet numerum quemcunque; e

$$dz = Pdx - \frac{nPxda}{a}.$$

Debebit ergo P talis esse quantitas, quae $dx - \frac{nxda}{a}$, si in id
reddat integrabile. Fit autem $dx - \frac{nxda}{a}$ integrabile, si ducatur
enim erit $\frac{x}{a^n}$. Quare generaliter erit

$$P = \frac{1}{a^n} f\left(\frac{x}{a^n} + c\right).$$

$$Q = -\frac{nx}{a^{n+1}} f\left(\frac{x}{a^n} + c\right).$$

telligitur etiam, si fuerit

$$P = X + \frac{1}{a^n} f\left(\frac{x}{a^n} + c\right),$$

ro quoque generalius

$$Q = A - \frac{nx}{a^{n+1}} f\left(\frac{x}{a^n} + c\right).$$

bi ut ante et in posterum semper f denotat functionem quamcunque qu
atis sequentis. At A est functio quaecunque ipsius a, et X functio q
nque ipsius x tantum.

10. Quo igitur dignosci queat, an datus quispiam valor ipsius P
rmula inventa contineatur, poni debebit $a = b^{\frac{1}{n}}$, quo facto videndum
Pb fiat functio ipsarum b et x nullius dimensionis, vel an prodeat ag
tum ex functione quadam ipsius x tantum et tali functione. Quod si
ehenderetur, habebit P proprietatem requisitam eritque Q aequale
si functioni in $-\frac{nx}{a}$ ductae una cum functione quacunque ipsius A.
niversum autem notandum est quantitatem P functione ipsius x ut X, e
nctione ipsius a ut A posse augeri. Nam si fuerit

$$dz = Pdx + Qda$$

quatio modularis, talis quoque erit aequatio

$$dz = Pdx + Xdx + Qda + Ada.$$

osito enim du loco $dz - Xdx - Ada$ habebitur $du = Pdx + Qda$, q
m priore prorsus congruit. Hanc ob rem superfluum foret in posterum
lorem ipsius Q assumtum functionem A ipsius a adiicere. Quare h
pparentem generalitatem negligemus.

11. Sit nunc $Q = PE$ denotante E functionem quamcunque ipsiu
rit itaque

$$dz = Pdx + PEda$$

$$dz = Pdx + Ada \qquad \frac{}{a}$$

seu

$$dz - Ada = Pdx - \frac{Pxda}{a}.$$

In qua aequatione cum $dz - Ada$ sit integrabile, debebit Pdx
esse integrabile. Hoc autem per praecedentem operatio
$P = \frac{1}{a} f\left(\frac{x}{a} + c\right)$. Tum igitur erit

$$Q = A - \frac{x}{a^2} f\left(\frac{x}{a} + c\right).$$

Simili ratione intelligitur, si fuerit

$$P = X + \frac{1}{a} f\left(\frac{x}{a} + c\right),$$

denotante X functionem ipsius x tantum, fore

$$Q = A - \frac{x}{a^2} f\left(\frac{x}{a} + c\right),$$

ubi ut ante $f\left(\frac{x}{a} + c\right)$ exprimit functionem quamcunque ipsaru
dimensionis.

9. Sit $Q = -\frac{nPx}{a}$, ubi n indicet numerum quemcunque;

$$dz = Pdx - \frac{nPxda}{a}.$$

Debebit ergo P talis esse quantitas, quae $dx - \frac{nxda}{a}$, si in
reddat integrabile. Fit autem $dx - \frac{nxda}{a}$ integrabile, si ducatu
enim erit $\frac{x}{a^n}$. Quare generaliter erit

$$P = \frac{1}{a^n} f\left(\frac{x}{a^n} + c\right).$$

elligitur etiam, si fuerit

$$P = X + \frac{1}{a^n} f\left(\frac{x}{a^n} + c\right),$$

quoque generalius

$$Q = A - \frac{nx}{a^{n+1}} f\left(\frac{x}{a^n} + c\right).$$

ut ante et in posterum semper f denotat functionem quamcunque q
tis sequentis. At A est functio quaecunque ipsius a, et X functio q
que ipsius x tantum.

10. Quo igitur dignosci queat, an datus quispiam valor ipsius
mula inventa contineatur, poni debebit $a = b^{\frac{1}{n}}$, quo facto videndum
Pb fiat functio ipsarum b et x nullius dimensionis, vel an prodeat ag
um ex functione quadam ipsius x tantum et tali functione. Quod si
hendetur, habebit P proprietatem requisitam eritque Q aequale
functioni in $-\frac{nx}{a}$ ductae una cum functione quaecunque ipsius A.
versum autem notandum est quantitatem P functione ipsius x ut X,
ctione ipsius a ut A posse augeri. Nam si fuerit

$$dz = Pdx + Qda$$

uatio modularis, talis quoque erit aequatio

$$dz = Pdx + Xdx + Qda + Ada.$$

ito enim du loco $dz - Xdx - Ada$ habebitur $du = Pdx + Qda$,
n priore prorsus congruit. Hanc ob rem superfluum foret in posterum
rem ipsius Q assumtum functionem A ipsius a adiicere. Quare
arentem generalitatem negligemus.

11. Sit nunc $Q = PE$ denotante E functionem quamcunque ipsi
t itaque

$$dz = Pdx + PEda$$

Sive si ponatur $\int E da = A$ fueritque $P = f(x + A)$, erit

$$Q = \frac{dA}{da} f(x + A).$$

Num autem datus ipsius P valor in hac formula contine
investigandum: ponatur $x = y - A$ et quaeratur, an pro
functio ipsius a et constantium, ut P fiat functio solius y et
modulus a non amplius ingrediatur.

12. Ponamus esse $Q = PY$, ubi Y sit functio q
modulum a non involvens. Quo posito erit

$$dz = Pdx + PYda$$

et P talis functio, quae efficiat $dx + Yda$ integrabile. Posit

$$z = \int \frac{dx}{Y} + a = X + a,$$

si ponatur $\int \frac{dx}{Y} = X$. Quamobrem erit

$$P = \frac{1}{Y} f(X + a).$$

Quoties ergo P huiusmodi habuerit valorem, erit semper

13. Sit nunc generalius positum $Q = PEY$, erit

$$dz = Pdx + PEYda,$$

ubi ut ante E denotat functionem ipsius a, Y vero ipsius
si fuerit $P = \frac{1}{Y}$, formulam istam differentialem effici in
enim

$$z = \int \frac{dx}{Y} + \int E da, \text{ seu } z = X + A$$

posito $\int \frac{dx}{Y} = X$. Quamobrem erit

$\ldots f(X + A) = \frac{\ldots}{dx} f(X + A)$ [illegible]

casibus fiet

$$Q = \frac{dA}{da} f(X + A).$$

ıduntur in his formulis etiam logarithmici ipsarum A et X valores,

$$X = lT \text{ et } A = -lF,$$

$$P = \frac{dT}{Tdx} f \frac{T}{F} \text{ et } Q = \frac{-dF}{Fda} f \frac{T}{F}.$$

erspicitur igitur omnes has formulas locum habere, si aequatio ‘uerit vel

$$dz = dX\, f(X + A) \text{ vel } dz = \frac{dX}{X} f \frac{X}{A}.$$

go aequatio proposita ad has formas poterit reduci, substituendis ctione quacunque ipsius x et A pro functione quacunque ipsius a, atio modularis poterit exhiberi: erit enim priore casu

$$dz = dX f(X + A) + dA f(X + A),$$

re vero casu

$$dz = \frac{dX}{X} f \frac{X}{A} - \frac{dA}{A} f \frac{X}{A}.$$

idem in his universalibus exemplis facile perspicitur, in specialioribus difficilius. Quocirca maximum positum erit subsidium in redu- ›us particularibus ad has generales formas, id quod, si quidem talis ri potest, non difficulter praestatur.

ponatur $Q = PR$, designante R functionem quamcunque ipsarum

$$dz = Pdx + PRda.$$

ndum nunc valorem ipsius P, sumatur formula $dx + Rda$, seu $x + Rda = 0$ consideretur et quaeratur, quomodo indeterminatae invicem possint separari, seu quod idem est, per quamnam quan-

$Q = RS/T$. Haec operatio latissime patet et omnes casus complec
Q cognitum et a z non pendentem habet valorem.

16. Progrediamur autem ulterius et in eos ipsius P valores
in quibus Q non solum a P sed etiam a $\int Pdx$ seu a z pendet. Po
primo

$$Q = \frac{nz}{a} - \frac{Px}{a},$$

denotante n numerum quemcunque. Erit ergo

$$dz = Pdx + \frac{nzda}{a} - \frac{Pxda}{a},$$

seu

$$dz - \frac{nzda}{a} = Pdx - \frac{Pxda}{a}.$$

Multiplicetur utrinque per $\frac{1}{a^n}$, quo prodeat haec aequatio

$$\frac{dz}{a^n} - \frac{nzda}{a^{n+1}} = \frac{Pdx}{a^n} - \frac{Pxda}{a^{n+1}},$$

in qua prius membrum est integrabile. Debebit ergo etiam inte
alterum membrum

$$\frac{Pdx}{a^n} - \frac{Pxda}{a^{n+1}},$$

ex quo idoneus ipsius P valor est quaerendus. Evenit hoc, si P
enim integrale $\frac{x}{a} + c$. Quare erit universaliter

$$P = a^{n-1} f\left(\frac{x}{a} + c\right),$$

id quod contingit, si $\frac{P}{a^{n-1}}$ est functio ipsarum a et x nullius dimen
functio ipsarum a et x dimensionum $n - 1$. Hoc igitur casu est

$$nz = Px + Qa,$$

ut in superiore dissertatione ostendimus [p. 46].

que

$$dz - \frac{nzda}{a} = Pdx + PEYda$$

$$\frac{dz}{a^n} - \frac{nzda}{a^{n+1}} = \frac{Pdx}{a^n} + \frac{PEYda}{a^n}.$$

rem P ita debet accommodari, ut

$$\frac{dx + EYda}{a^n}$$

ultiplicatum evadat integrabile. Fit hoc autem, si $P = \frac{a^n}{Y}$, quo casu

est $\int \frac{dx}{Y} + \int Eda$ seu $X + A$ posito $\int \frac{dx}{Y} = X$ et $\int Eda = A$.

debebit esse

$$P = \frac{a^n dX}{dx} f(X + A),$$

casibus erit

$$Q = \frac{a^n dA}{da} f(X + A) + \frac{nz}{a}.$$

l a logarithmis pendeant, prodibit P huius valoris

$$\frac{a^n dX}{Xdx} f \frac{X}{A},$$

ndebit

$$Q = \frac{nz}{a} - \frac{a^n dA}{Ada} f \frac{X}{A}.$$

Si ponatur $Q = Fz + PEY$, et F et E functiones sint ipsius a,

psius x, tum erit

$$dz - Fzda = Pdx + PEYda.$$

$\int Fda = lB$, ita ut B sit functio ipsius a, et dividatur per B, habebitur

$$\frac{dz}{B} - \frac{zdB}{B^2} = \frac{Pdx}{B} + \frac{PEYda}{B}.$$

Cum igitur prius membrum sit integrabile, et alterum tale hoc, si $P = \frac{B}{Y}$, tumque erit integrale

$$\int \frac{dx}{Y} + \int Eda \text{ seu } X + A.$$

Quocirca erit ipsius P valor quaesitus

$$\frac{BdX}{dx} f(X + A),$$

Q vero erit

$$\frac{zdB}{Bda} + \frac{BdA}{da} f(X + A).$$

Perspicitur quoque, si fuerit

$$P = \frac{BdX}{Xdx} f \frac{X}{A}, \text{ fore } Q = \frac{zdB}{Bda} - \frac{BdA}{Ada} f \frac{X}{A}$$

19. Latissime patebit solutio, si ponatur

$$Q = Fz + PR$$

et R fuerit functio ipsarum a et x. Erit enim

$$dz - Fzda = Pdx + PRda.$$

Posito $\int Fda = lB$ dividatur per B, habebitur

$$\frac{dz}{B} - \frac{zdB}{B^2} = \frac{P}{B}(dx + Rda).$$

Sit iam S functio efficiens $dx + Rda$ integrabile sitque

$$\int (Sdx + SRda) = T.$$

Quo invento erit $P = BSfT$, huic respondet $Q = \frac{zdB}{Bda} + $

20. Possunt praeterea plures huiusmodi valores ipsius modo multo latius extendi, ut, si ponatur

$$P = \frac{BdX}{dx} f(X + A) + \frac{BdY}{dx} f(Y + E),$$

erit

$$Q = \frac{zdB}{Bda} + \frac{BdA}{da} f(X + A) + \frac{BdE}{da} f(Y +$$

venitur. Quamobrem his expeditis pergo ad eos casus investigandos, in q
quatio modularis primi gradus differentialis non datur, sed qui tamc
quationem modularem differentio-differentialem perducuntur.

21. Si igitur Q neque algebraice per a et x neque per z potest expri
vestigandi sunt casus, quibus differentiale ipsius Q poterit exhiberi. Est a

$$Q = \frac{dz - Pdx}{da},$$

go

$$dQ = d \cdot \frac{dz - Pdx}{da}.$$

uare si differentiale ipsius Q vel per sola a et x vel per haec et Q vel c
nul per z poterit exprimi, habebitur aequatio modularis, quae erit diff
lis secundi gradus. Ostensum autem est superiore dissertatione {p. 3
natur

$$dP = Ldx + Mda,$$

e

$$dQ = Mdx + Nda,$$

ut haec differentialia communem litteram M involvant. Quia autem ex
etiam M datur, nil aliud requiritur, nisi ut N determinetur. Quamobr
inquiremus casus, quibus N vel algebraice, vel per Q, vel per Q et z ex
test. Tum enim habebitur aequatio modularis

$$Mdx + Nda = d \cdot \frac{dz - Pdx}{da},$$

sito in N loco Q eius valore $\frac{dz - Pdx}{da}$.

22. Ex praecedentibus satis intelligitur, si N per sola a et x determin
e

$$M = \frac{dX}{dx} f(X + A) \text{ et } N = \frac{dA}{da} f(X + A),$$

$$M = V + \frac{dX}{dx} f(X + A) \text{ et } N = I + \frac{dA}{da} f(X + A)$$

$$V + \frac{dX}{dx} f(X + A)$$

contineatur. Quod si fuerit compertum et X et A et V definitae,

$$Vdx + dX f(X + A) + Ida + dA f(X + A) = d \cdot \frac{dz -}{}$$

aequatio modularis desiderata. Notandum est in posterum $\frac{dX}{dx} f(X + A)$ poni posse aggregatum ex quotvis huiusmodi for

$$\frac{dX}{dx} f(X + A) + \frac{dY}{dx} f(Y + B) + \text{etc.}$$

At loco $\frac{dA}{da} f(X + A)$ tunc poni debebit

$$\frac{dA}{da} f(X + A) + \frac{dB}{da} f(Y + B) + \text{etc.}$$

Hoc igitur monito in posterum tantum unica formula $\frac{dX}{dx} f($

respondente $\frac{dA}{da} f(X + A)$ utemur.

23. Pendeat N simul etiam a Q sitque

$$N = R + DQ,$$

ubi D sit functio ipsius a, et R functio ipsarum a et x ex conditio
tibus determinanda. Erit igitur

$$dQ - DQda = Mdx + Rda,$$

sit

$$Dda = \frac{dH}{H}$$

et dividatur utrinque per H, prodibit

$$\frac{dQ}{H} - \frac{QdH}{H^2} = \frac{Mdx + Rda}{H}.$$

— est efficiendum. Fiet igitur per praecedentem methodum

$$M = \frac{HdX}{dx} f(X + A) \text{ et } R = \frac{HdA}{da} f(X + A).$$

in exemplo quopiam proposito ex P reperiatur M talis valoris, erit

$$N = \frac{HdA}{da} f(X + A) + \frac{dH}{Hda^2}(dz - Pdx)$$

$\frac{H}{da}$ loco D et $\frac{dz - Pdx}{da}$ loco Q. Atque hinc in promptu erit aequatio

.

i N non a Q sed a z pendeat, ita ut sit

$$N = R + Cz,$$

C functionem ipsius a quamcunque, erit

$$dQ - Czda = Mdx + Rda.$$

st

$$dz - Qda = Pdx,$$

uius multiplum

$$Fdz - QFda = PFdx,$$

F functione ipsius a, quo facto orietur aequatio

$$dQ - QFda + Fdz - Czda = (M + PF)dx + Rda.$$

$$Fda = \frac{dB}{B} \text{ et } \frac{Cda}{F} = \frac{dG}{G},$$

$$F = \frac{dB}{Bda} \text{ et } C = \frac{dBdG}{BGda^2}.$$

m itaque est $dQ - QFda$ integrabile reddi, si dividatur per B seu tur per $\frac{1}{B}$, $Fdz - Czda$ autem fit integrabile, si multiplicetur per $\frac{1}{FG}$. o idem factor summam horum differentialium reddat integrabilem, se $FG = B$ seu $\frac{GdB}{Bda} = B$, unde fiet $G = \frac{B^2da}{dB}$. Hanc ob rem alterum embrum per B divisum est integrabile efficiendum scilicet

$$\frac{(M + PF)dx + Rda}{B}.$$

et

$$M + PF = \frac{BdX}{dx}\, l(X + A) = M + \frac{PdB}{Bda}.$$

Investigari igitur debet proposito exemplo, an loco A, B et X ta
inveniri queant, quae exhibeant formulam

$$\frac{BdX}{dx}\, l(X + A)$$

aequalem ipsi

$$M + \frac{PdB}{Bda}.$$

Hisque inventis erit

$$N = \frac{BdA}{da}\, l(X + A) + \frac{zdBdG}{BGda^2}$$

existente $G = \frac{B^2 da}{dB}$, qui valor in aequatione

$$Mdx + Nda = d \cdot \frac{dz - Pdx}{da}$$

substitutus dabit aequationem modularem.

25. Sit nunc generalissime

$$N = R + DQ + Cz,$$

tenentibus R, D et C iisdem quibus ante valoribus. Erit ergo

$$dQ - DQda - Czda = Mdx + Rda;$$

addatur ad hanc aequatio

$$Fdz - FQda = PFdx,$$

quo habeatur

$$dQ - DQda - FQda + Fdz - Czda = (M + PF)dx +$$

Positis autem ut ante

$$Dda = \frac{dH}{H},\ Fda = \frac{dB}{B},\ \text{et}\ \frac{Cda}{F} = \frac{dG}{G},$$

m est integrabile, fiet ergo facto $HB = E$

$$R = \frac{EdA}{da} f(X + A) \text{ et } M + PF = \frac{EdX}{dx} f(X + A).$$

in casu proposito A, X, E et F, si fieri potest, ita debent definiri, ut $+ A)$ aequale fiat ipsi $M + PF$. Hocque invento erit

$$N = \frac{EdA}{da} f(X + A) + \frac{dH}{Hda^2}(dz - Pdx) + \frac{FzdG}{Gda},$$

uatio modularis reperitur.

t si nequidem differentialis secundi gradus aequatio modularis ob-erit, ad differentialia tertii gradus erit procedendum. Fiet ergo

$$N = \frac{d\left(\frac{dz - Pdx}{da}\right) - Mdx}{da}$$

e posito $dN = sdx + tda$ erit

$$sdx + tda = d\left(\frac{d\left(\frac{dz - Pdx}{da}\right) - Mdx}{da}\right).$$

em s ex M, cum sit sda differentiale ipsius M, quod prodit, si x pona-ans. Quamobrem t tantum debebit investigari. Sit ergo

$$t = R + EN + DQ + Cz$$

$$dN - ENda - DQda - Czda = sdx + Rda.$$

addantur horum multipla ad illam aequationem, ut prodeat h

$$dN - ENda - FNda + FdQ - DQda - GQda + Gdz -$$
$$(s + MF + PG)dx + Rda.$$

Sit

$$Eda + Fda = \frac{df}{f}, \quad \frac{Dda + Gda}{F} = \frac{dg}{g} \text{ et } \frac{Cda}{G} = \frac{dh}{h}$$

fiatque

$$f = Fg = Gh.$$

Quo facto aequationis inventae prius membrum fit integrabile
hanc ob rem et

$$\frac{(s + MF + PG)dx + Rda}{f}$$

efficiendum est integrabile. Ponendum igitur est

$$R = \frac{fdA}{da} f(X + A)$$

et

$$s + MF + PG = \frac{fdX}{dx} f(X + A).$$

In aequatione ergo proposita, quia s et M ex P dantur, debent
ex hac aequatione determinari. Quo facto sumatur $g = \frac{f}{F}$ et h

$$C = \frac{Gdh}{hda} \text{ et } D = \frac{Fdg}{gda} - G \text{ et } E = \frac{df}{fda} - F.$$

Atque ex his cognita erit aequatio

$$t = R + EN + DQ + Cz,$$

ex qua aequatio modularis facile conflatur. Simili modo ex
quomodo pro altioribus differentialium gradibus operatio deb
ad aequationes modulares perveniatur.

27. In compendium nunc, quae hactenus tradidimus,
quo facilius quaevis aequatio proposita reduci queat, tum qu

$$dP = Mda, \quad dM = pda, \quad dp = rda \text{ etc.}$$

$$Q = \frac{dz - Pdx}{da}, \quad N = \frac{dQ - Mdx}{da},$$

$$q = \frac{dN - pdx}{da} \text{ et } s = \frac{dq - rdx}{da} \text{ etc.,}$$

V et dq etc. sunt differentialia ipsorum Q, N et q, quae ex valoribus

$$\frac{dz - Pdx}{da}, \quad \frac{dQ - Mdx}{da} \quad \text{et} \quad \frac{dq - rdx}{da}$$

r positis a, x et z variabilibus. Hanc igitur ob rem cognitae erunt e. ex solo P, ex his vero habebuntur Q, N, q etc. Sint praeterea , E, F etc. functiones ipsius a et constantium, et X, Y etc. functiones n involventes a.

His praemissis si fuerit P talis functio ipsius x et a, ut BP com- ur [§ 18] in hac forma

$$\frac{dX}{dx} f(X + A)$$

n huiusmodi formularum aggregato, semper dari poterit aequatio differentialis primi gradus. Namque erit

$$PdAdx = z\frac{dBdX}{B} + QdadX$$

$$BPdAdx = zdBdX + BQdadX.$$

atio ob datum Q est modularis respondens aequationi propositae.

einde si P talis sit functio ipsarum a et x, ut

$$BP + CM$$

eri possit [§ 24]

$$\frac{dX}{dx} f(X + A)$$

$$BPdAdx + CMdAdx = zdBdX + BQdadX + QdCdX +$$

Quae est aequatio modularis quaesita, et involvit differentialia se
quia eam littera N ingreditur, quae per dQ ideoque per ddz
determinatur.

30. At si fuerit

$$BP + CM + Dp$$

aequalis huic formulae

$$\frac{dX}{dx} \int (X + A)$$

vel aggregato quotcunque huiusmodi formularum, aequatio m
differentialis tertii gradus, prodibit enim ista aequatio

$$\begin{aligned} BPdAdx + CMdAdx + DpdAdx = zdBdX + BQdadX \\ + CNdadX + NdDdX + DqdadX. \end{aligned}$$

Quemadmodum ex ante traditis colligere licet, si modo quantitate
pendentes ad has formulas accommodantur.

31. Simili modo ad altiora differentialia progressus faci
Nam si

$$BP + CM + Dp + Er$$

aequetur formulae

$$\frac{dX}{dx} \int (X + A)$$

vel talium plurium formularum aggregato, orietur aequatio modu

$$\begin{aligned} BPdAdx + CMdAdx + DpdAdx + ErdAdx = zdBdX \\ + QdCdX + CNdadX + NdDdX + DqdadX + qdEdX \end{aligned}$$

quae erit differentialis quarti gradus. Atque hoc modo quousqu
operationes facile continuantur ex sola allatarum inspectione.

... posita erit in cognoscenda functione P, an in his expositis generi... ontineatur et in quonam genere. Etiam si enim generales ipsius P valores, ... x assumtis formulis obtinentur, nihil difficultatis in se habere videan... amen exemplis particularibus propositis accommodatio saepissime erit d... llima. Cuius rei ratio nequaquam methodo traditae est tribuenda, sed imp... ectae functionum cognitioni, quae adhuc habetur. Quamobrem non sol... hoc negotio, sed in plurimis etiam aliis casibus maxime utile foret, si fu... onum doctrina magis perficeretur et excoleretur.

33. Quantum quidem mihi hac de re meditari licuit, eximium subsidi... veni, si P statim ad huiusmodi formam $\frac{dX}{dx} / (X + A)$ vel huiusmodi form... m aggregatum reducatur, id quod sequenti modo facillime praestat... rima aequatio proposita non constituatur inter z et x, sed inter z et y, ita... quatio ad modularem perducenda sit $dz = T dy$, existente T functi... sius y et moduli a. Tum accipiatur pro x talis functio ipsarum a et y, q... ansmutet T in functionem ipsarum a et x contentam in formula $/ (X +$ ... l pluribus huic similibus earumque multiplis, in quibus X est functio ips... tantum, et A ipsius a. Hoc igitur facto prodeat aequatio $dz = S dx / (X +$ ... i S sit quantitas tam simplex quam fieri potest. Quare P erit $S / (X +$ ... eoque cum M, p etc. coniuncta facilius cum generalibus formulis comparat... venta autem hoc modo aequatione modulari, valor ipsius x in a et y assu... s ubique loco x, loco dx autem differentiale huius valoris positis a e... riabilibus substituatur. Quo facto habebitur aequatio modularis in... y et z, quae quaerebatur.

34. Ad pleniorem quidem methodi hactenus traditae cognition... aximam lucem afferrent exempla et problemata, quorum solutio ist... ethodum requirit. Sed quia ipsorum problematum dignitas peculiar... actationem postulat, in aliud tempus[1]), ne hoc tempore nimis sim longus, e... ffero.

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1) Vide L. EULERI Commentationem 52: *Solutio problematum rectificationem ellipsis requi...m*, Comment. acad. sc. Petrop. 8 (1736), 1741, p. 86. Vide quoque notam p. 16 huius volum... nec non Commentationes 11, 31, 70, 274 et *Institutiones calculi integralis*, vol. II, § 1016—1... 69—1078. LEONHARDI EULERI *Opera omnia*, series I, vol. 20, 22, 12. H. I...

INVESTIGATIO BINARUM CURVA
QUARUM ARCUS EIDEM ABSCISSAE RE
SUMMAM ALGEBRAICAM CONSTI

Commentatio 48 indicis ENESTROEMIANI
Commentarii academiae scientiarum Petropolitanae 8 (1736), 1

1. Problema, cuius solutionem hac dissertatione
sequentes continet conditiones. Requiruntur in eo I. duae
quarum II. neutra sit rectificabilis, quae tamen ita debe
ut duo arcus III. eidem abscissae respondentes IV. su
algebraicam. Harum quatuor conditionum quaecunque o
solutu admodum facile, omnibus autem satisfacere maxi
Prima quidem conditione omissa, si admittantur curv
reliquis conditionibus facile satisfiet. Si secunda omitta
curvae algebraicae et rectificabiles problemati satisfaci
neglecta difficilior est solutio, sed tamen ex iis, quae Celeb
et BERNOULLIUS[2]) de reductione quadraturarum ad recti
algebraicarum dederunt, solutio facile deducitur. Quarta
omittatur, ne quidem problema erit, cum omnes curv
rectificabiles reliquis conditionibus satisfaciant.

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1) IAC. HERMANN (1678—1733), *Solutio propria duorum problematu*
Erudit. 1719 *Mens. Aug. a se propositorum*, Acta erud. 1723, p. 171.

2) IOH. BERNOULLI (1667—1748), *Constructio facilis curvae recessu*
per rectificationem curvae algebraicae, Acta erud. 1694, p. 394. *Theorem*
linearum curvarum inserviens. Acta erud. 1698, p. 462. *Methodus inveniendi c*
non quadrabiles, habentes tamen numerum determinatum spatiorum absolute
Suppl. t. VIII, 1724, p. 380. *Methodus commoda et naturalis reducendi quadr*
vis gradus ad longitudines curvarum algebraicarum. Acta erud. 1724, p. 356
et 249, t. II, p. 315 et 582.

2. Ad generalem huius problematis solutionem utor formulis, q
tati Viri Celeb. dederunt pro curvis vel rectificabilibus, vel quarum rectific
data quadratura pendet. His enim formulis effici potest, ut curvae
gebraicae, ut sint non rectificabiles, atque ut arcuum summa sit rectificab
onstrabo vero etiam, quomodo abscissae aequales reddi possint. Quo fa
mnibus conditionibus erit satisfactum, atque problema generaliter solut
am late enim istae formulae patent, ut, nisi praeter necessitatem restri
dhibeatur, omnes omnino curvas problemati satisfacientes exhibere debea

3. Designatis igitur curvis quaesitis per litteras A et B, erit ex
rmulis[1])

in Curva A		in Curva B	
abscissa	$\dfrac{(dP^2-dQ^2)^{\frac{3}{2}}}{dQddP-dPddQ}$	abscissa	$\dfrac{(dp^2-dq^2)^{\frac{3}{2}}}{dqddp-dpddq}$
applicata	$P+\dfrac{dQ(dP^2-dQ^2)}{dQddP-dPddQ}$	applicata	$p+\dfrac{dq(dp^2-dq^2)}{dqddp-dpddq}$
arcus	$Q+\dfrac{dP(dP^2-dQ^2)}{dQddP-dPddQ}$	arcus	$q+\dfrac{dp(dp^2-dq^2)}{dqddp-dpddq}$

s formulis iam obtinetur, quod alias maximam pareret difficultatem,
bae curvae sint algebraicae, si modo P ponatur quantitas algebraic
eindo rectificabiles non erunt, si Q et q quantitates transcendentes involvu
rtio arcuum summa erit rectificabilis, si $Q+q$ fuerit quantitas algebra
amsi Q et q seorsim tales non sint. Cum autem his conditionibus fu
isfactum, abscissae inter se aequales sunt efficiendae.

4. Efficiamus primo abscissas inter se aequales eritque

$$\frac{(dP^2-dQ^2)^{\frac{3}{2}}}{dQddP-dPddQ}=\frac{(dp^2-dq^2)^{\frac{3}{2}}}{dqddp-dpddq}.$$

at ad hoc praestandum $dQ=RdP$ et $dq=rdp$. Quo posito habebi

$$\frac{(1-R^2)^{\frac{3}{2}}dP}{-dR}=\frac{(1-r^2)^{\frac{3}{2}}dp}{-dr},$$

1) Confer Commentationem 245 huius voluminis § 70, Solutio I, p. 280. H. I

2) p, dq, dQ quoque ponuntur quantitates algebraicae. H.

$$dP = \frac{\quad}{(1-R^2)^{\frac{3}{2}} dr},$$

quod differentiale, quia P debet esse quantitas algebraica, es
reddendum. Sunt autem R et r quantitates algebraicae, ob cu
algebraicas, quare et

$$\frac{(1-r^2)^{\frac{3}{2}} dR}{(1-R^2)^{\frac{3}{2}} dr}$$

erit quantitas algebraica. Posito igitur brevitatis gratia

$$\frac{(1-r^2)^{\frac{3}{2}} dR}{(1-R^2)^{\frac{3}{2}} dr} = T, \text{ erit } dP = T dp, \text{ seu } P = Tp - \int p dT$$

Quo ergo P sit quantitas algebraica, facio $\int p dT = N$, eritque

$$p = \frac{dN}{dT} \text{ et } P = \frac{TdN}{dT} - N.$$

5. Hac igitur ratione iam assecuti sumus valores algebraico
quibus substitutis utriusque curvae abscissae fiunt aequales. Pra
ipsae erunt algebraicae, si modo R, r et N fuerint tales. Sed quo ar
fiat quoque algebraica, Q et q ita determinari debent, ut $Q + q$
algebraica. Est vero

$$Q + q = \int R dP + \int r dp = RP + rp - \int P dR - \int p$$

Ponatur igitur

$$\int P dR + \int p dr = M,$$

eritque

$$P = \frac{dM - p dr}{dR}$$

atque

$$Q + q = RP + rp - M.$$

6. Cum autem iam supra inventum sit

$$p = \frac{dN}{dT} \text{ et } P = \frac{TdN}{dT} - N,$$

ntur hi valores in aequatione

$$PdR + pdr = dM.$$

o prodibit

$$\frac{TdNdR}{dT} - NdR + \frac{dNdr}{dT} = dM.$$

o M est quantitas algebraica, oportet ut hic ipsius dM valor possit
Integratione autem instituta prodit

$$M = \frac{TNdR}{dT} + \frac{Ndr}{dT} - \int N\left(dR + d\cdot\frac{TdR}{dT} + d\cdot\frac{dr}{dT}\right).$$

e hoc integrale $= u$, ideoque debet esse

$$N = \frac{du}{dR + d\cdot\frac{TdR}{dT} + d\cdot\frac{dr}{dT}};$$

, r et u quantitates quaecunque algebraicae accipi poterunt.

umtis igitur pro R, r et u functionibus quibuscunque indeterminatae z,
uoque T in z, cum sit

$$T = \frac{(1-r^2)^{\frac{3}{2}}dR}{(1-R^2)^{\frac{3}{2}}dr}.$$

postrema aequatione reperietur quoque N in z. Inventa autem N

$$M = \frac{TNdR}{dT} + \frac{Ndr}{dT} - u.$$

modo dabuntur P et p per z ex aequationibus

$$P = \frac{TdN}{dT} - N \text{ et } p = \frac{dN}{dT}.$$

habebitur

$$Q + q = RP + rp - M.$$

lis igitur determinationibus consecuti sumus primo, ut curvarum
bscissae sint aequales, secundo ut utraque curva sit algebraica, et
arcuum summa sit rectificabilis. Quare videamus, an quoque con-

et q proveniant, cavendum tantum est, ne $\frac{drdN}{dT}$ fiat integ
$dq = rdp$, erit

$$q = rp - \int pdr = rp - \int \frac{drdN}{dT}$$

atque

$$Q = RP - M + \int \frac{drdN}{dT} \cdot$$

9. Quo autem appareat, quomodo evitari possit $\frac{drdN}{dT}$, problema etiam quinta adiecta conditione solvan curva utraque non solum sit irrectificabilis, sed etiam ut a data pendeat quadratura, puta a $\int Zdz$. Ad hoc igitur $\int \frac{drdN}{dT}$ ad $\int Zdz$ reduci. Est vero

$$\int \frac{drdN}{dT} = \frac{Ndr}{dT} - \int Nd \cdot \frac{dr}{dT} = \frac{Ndr}{dT} - \int \frac{du}{dR + d \cdot}$$

posito loco N eius valore § 6 invento.

10. Ponatur brevitatis gratia

$$\frac{d \cdot \frac{dr}{dT}}{dR + d \cdot \frac{TdR}{dT} + d \cdot \frac{dr}{dT}} = S,$$

quae ergo quantitas ex solis r et R est composita. Quare

$$\int \frac{drdN}{dT} = \frac{Ndr}{dT} - \int Sdu = \frac{Ndr}{dT} - Su + \int$$

Fiat igitur

$$\int udS = \int Zdz,$$

unde reperitur

$$u = \frac{Zdz}{dS} \cdot$$

$$\int \frac{dr\,dN}{dT} = \frac{N\,dr}{dT} - \frac{S\,Z\,dz}{dS} + \int Z\,dz.$$

ide cum eadem quadratura infinitis modis possit exhiberi, non solum trarios ipsarum R et r valores varietas infinita obtinetur, sed etiam umeros ipsius u valores; quibus tamen omnibus efficitur, ut curv entarum omnium rectificatio a quadratura proposita $\int Z\,dz$ pendeat[1]).

11. Hac igitur ratione innumerabilibus modis solvi problema non so nitio proposueram, sed adiecta insuper conditione pendentiae rectificat varum inveniendarum a data quadratura. Problema igitur hac tum ita est proponendum. Duas invenire curvas algebraicas, qu usque rectificatio a data pendeat quadratura, duorum autem arcuum c cissae respondentium summa sit rectificabilis.

12. Ipsae autem curvae quaesitae determinabuntur ex assumtis ris R et r valoribus algebraicis atque ex u propositam quadraturam i ente. Ex his enim reperiuntur P et p, quibus inventis erit curvae A

$$\text{abscissa} = \frac{(1-R^2)^{\frac{3}{2}}dP}{-dR} \text{ et applicata} = P + \frac{R\,dP(1-R^2)}{-dR}.$$

rius vero curvae B

$$\text{abscissa} = \frac{(1-r^2)^{\frac{3}{2}}dp}{-dr},$$

aequalis erit illius abscissae; at

$$\text{applicata erit} = p + \frac{r\,dp(1-r^2)}{-dr}.$$

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1) Cf. Commentationem 245 huius voluminis. Vide quoque Commentationes 622, 650 782, 817. *Specimen singulare analyseos infinitorum indeterminatae. Nova acta acad. sc. Pet* p. 47. *De formulis differentialibus, quae per duas pluresve quantitates datas multiplicatae abiles. Nova acta acad. sc. Petrop.* 7, 1793, p. 3. *Solutio problematis ad analysin infin rminatorum referendi. Mémoires de l'acad. des sc. de St. Pétersb.* 11, 1830, p. 92. *De in algebraicis, quarum longitudo indefinita arcui elliptico aequatur. Mémoires de l'acad. des étersb.* 11, 1830, p. 95. *De binis curvis algebraicis eadem rectificatione gaudentibus. Mémo . des sc. de St. Pétersb.* 11, 1830, p. 102. *De lineis curvis, quarum rectificatio per datam quadra uratur. Opera postuma* 1, 1862, p. 439. *LEONHARDI EULERI Opera omnia*, series I, vol. 23 et 21.

$$\frac{\quad}{-dR} + \int RdP,$$

et curvae B arcus eidem abscissae respondens erit

$$\frac{dp(1-r^2)}{-dr} + \int rdp.$$

Pendebit autem tam $\int RdP$ quam $\int rdp$ a $\int Zdz$; nihilo ta

$$\int RdP + \int rdp$$

algebraice poterit exhiberi.

13. Denique ex ipsa solutione satis intelligitur me
opera solvi posse problema, si non arcuum summa, se
debeat esse algebraica, vel etiam summa seu differentia q
plorum horum arcuum. Quamobrem superfluum foret
attingere. Ad institutum quidem plenius persequendu
exempla quaedam evolverentur, sed cum ad prolixiss
perveniendum, ea potius omitto aliisque investiganda reli

DE CONSTRUCTIONE AEQUATIONUM OPE MOTUS TRACTORII ALIISQUE AD METHODUM TANGENTIUM INVERSAM PERTINENTIBUS

Commentatio 51 indicis ENESTROEMIANI
Commentarii academiae scientiarum Petropolitanae 8 (1736), 1741, p. 66—85

Motu tractorio curvae lineae describuntur, dum filum datae longitudinis uno termino pondus annexum habens, altero termino in data linea recta sive curva protrahitur; atque ea linea curva, quam pondus motu suo

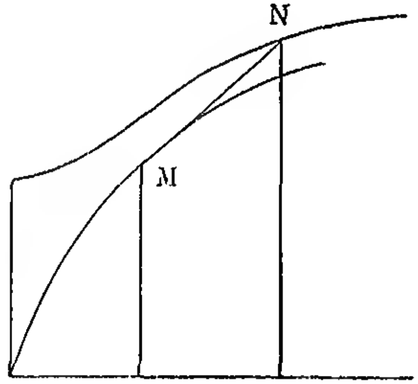

Fig. 1

describit, tractoria vocatur. Ut si (Fig. 1) filum BA in A pondere onustum in linea data BN protrahatur, linea AM, in qua alter terminus A movetur, erit curva tractoria. Huius curvae ista nota est proprietas, quod filum perpetuo positum sit in tangente curvae tractoriae; scilicet quando filum est in NM et hoc modo punctum M curvae tractoriae generat, erit recta

curva BN pro tractoria AM aequatio potest inveniri.

2. Ratio autem huius descriptionis ex mechanica
motus natura pendet. Movetur enim corpus semper in ea
protrahitur, si quidem quiescit; *atque hoc casu directio fili,*
est tangens curvae a corpore descriptae. At si corpus iam
directio a directione fili discrepabit. Quare quo motus
positionem fili incidat, oportet ut motus corpori iam impress
pereat. Ad hoc ergo obtinendum requiritur, ut haec d
super plano horizontali et satis aspero, illud quidem, ne
tionem immutet, hoc vero ut frictione omnis motus iam
Praeterea filum tardissimo protrahi debet, quo effectus fri
et corpus nihil de pristino motu retineat.

3. Si igitur hoc modo curva tractoria AM describat
proprietatem, ut ex quovis puncto M ducta tangens M
curvam BN sit datae magnitudinis. Ex quo perfacilis ori
curva tractoria AM inveniendi curvam BN, cuius illa est
fili longitudine. At ex data curva BN innumerabiles oriri
longitudine fili immutata, prout initio positio fili BA ad
inclinata. Longe autem difficilius est per calculum ex data
tractoriam AM quam ex tractoria AM data curvam BN.

4. Observavi autem geometricam constructionem trac
pendere a resolutione aequationis

$$ds + ssdz = Zdz,$$

denotante Z functionem quamcunque ipsius z. Quare e
constructu sit valde difficilis, quippe multo generalior qua

$$ds + ssdz = z^m dz,$$

quae a Com. Riccati[1]) quondam erat proposita, eius constr
motus attentionem meretur. Quae constructio cum sit p
simplex et facilis, operae pretium erit aequationis tam diffic
ad motum tractorium reduxisse.

1) Vide notam p. 17.

$$AP = x \text{ et } PM = y, \text{ sitque } dy = p\,dx;$$

nem fili vero AB vel MN pono $= b$. His positis erit[1])

$$\sqrt{(1 + pp)} : 1 = MN\,(b) : PQ\,(t - x),$$

$$\sqrt{(1 + pp)} : p = MN\,(b) : QN = PM\,(u - y).$$

nr fit

$$\frac{b}{\sqrt{(1 + pp)}} = t - x \text{ et } \frac{bp}{\sqrt{(1 + pp)}} = u - y,$$

his porro $pt - px = u - y$. Hanc postremam aequationem differen-
do $p\,dx$ loco dy, quo facto prodit

$$pdt + tdp - xdp = du$$

$$x = t + \frac{pdt}{dp} - \frac{du}{dp}.$$

ex prima aequatione

$$x = t - \frac{b}{\sqrt{(1 + pp)}},$$

inetur ista aequatio

$$du = pdt + \frac{bdp}{\sqrt{(1 + pp)}},$$

ae tantum insunt variabiles p et t, quia u per t datur.

Est autem p cotangens anguli MNQ posito sinu toto $= 1$, quare
atio ope motus tractorii resolvitur, per illum enim innotescet angulus
sque consequenter cotangens, cui aequalis est p. Ad irrationalitatem
llendam pono

$$\sqrt{(1 + pp)} = p + q \text{ seu } q = \sqrt{(1 + pp)} - p;$$

m $\sqrt{(1 + pp)}$ est cosecans anguli MNQ et p eius cotangens, erit
enta trigonometrica q tangens semissis anguli MNQ. Per hanc vero
onem est

---

[1]) $N\,(b)$ significat b esse longitudinem lineae MN. H. D.

atque

$$dp = \frac{-dq(1+qq)}{2qq}.$$

Hinc ergo erit $\frac{dp}{\sqrt{(1+pp)}} = \frac{-dq}{q}$, atque superior aequatio tr

$$2qdu = dt - qqdt - 2bdq.$$

7. Ad hanc aequationem ulterius reducendam pono

$$2bqdr + 2brdq = rdt - rqqdt;$$

in qua t et r a se mutuo pendent, quia t est $= AQ$, et blr

$$qr = s \text{ seu } q = \frac{s}{r},$$

erit

$$2bds = rdt - \frac{ssdt}{r}.$$

Sit nunc

$$\frac{dt}{r} = 2bdz \text{ et } rdt = 2bZdz,$$

erit

$$rr = Z \text{ et } r = \sqrt{Z}.$$

Praeterea est

$$dt^2 = 4b^2 Zdz^2 \text{ et } t = 2b\int dz\sqrt{Z}.$$

Per z igitur curva BN ita determinatur, ut sit

$$AQ = 2b\int dz\sqrt{Z} \text{ et } QN = \frac{b}{2}lZ.$$

Quia ergo curva BN datur, dabitur simul Z per z. Factis tutionibus habebitur

$$ds + ssdz = Zdz.$$

8. Proposita ergo aequatione

$$ds + ssdz = Zdz$$

valor ipsius s per z sequenti modo poterit definiri. Construatur cu
huiusmodi, ut sumta

$$\text{abscissa } AQ = 2b\int dz\sqrt{Z} \text{ sit applicata } QN = \frac{b}{2}lZ.$$

Tum filo longitudinis b secundum curvam BN protracto describatur t
AM. Deinde ducatur tangens MN, quae etiam ipso filo exhibebitu
secetque angulus MNQ, cuius dimidii tangens sit $= q$. Hoc facto erit

$$s = qr = q\sqrt{Z}.$$

9. Coordinatae autem AP et PM curvae tractoriae ita se habebu

$$AP = x = t - \frac{b}{\sqrt{(1+pp)}} = t - \frac{2bq}{1+qq}$$

et

$$y = u - \frac{bp}{\sqrt{(1+pp)}} = u - \frac{b(1-qq)}{1+qq}.$$

Quia autem est

$$t = 2b\int dz\sqrt{Z} \text{ et } u = \frac{b}{2}lZ, \text{ atque } q = \frac{s}{r} = \frac{s}{\sqrt{Z}},$$

erit[1])

$$x = 2b\int dz\sqrt{Z} - \frac{2bs\sqrt{Z}}{s^2+Z} \text{ et } y = \frac{b}{2}lZ + \frac{bs^2-bZ}{s^2+Z}.$$

Ex his iam aliae nascuntur constructiones aequationis

$$ds + ssdz = Zdz.$$

Per motum enim tractorium innotescunt coordinatae x et y curvae AM,
ex his erit[2])

$$\text{vel } s = \frac{\sqrt{Z}(t-x)}{u+b-y} \text{ vel } s = \frac{\sqrt{Z}(b-u+y)}{t-x}.$$

---

1) Editio princeps:

$$x = 2b\int dz\sqrt{Z} - \frac{2bs\sqrt{Z}}{s+Z} \text{ et } y = \frac{b}{2}lZ + \frac{bs-bZ}{s+Z}.$$

Correxit

2) Editio princeps:

$$s = \frac{Z(t-x)}{2b\sqrt{Z}-t+x}, \text{ vel } s = \frac{Z(b-u+y)}{b+u-y}.$$

Correxit

10. Aequatio vero inter x et y ex data ...
invenitur. Est enim ex aequationibus supra inven...

$$t = x + \frac{b}{\sqrt{(1+pp)}} = x + \frac{}{\sqrt{(d}}$$

et

$$u = y + \frac{bp}{\sqrt{(1+pp)}} = y + \frac{}{\sqrt{(d}}$$

Quare si in aequatione data inter t et u loco t et
prodibit aequatio inter x et y pro traetoria AM
tialis primi gradus, si aequatio inter t et u fuerit a
tione, quae plerumque fit maxime intricata, nihil
AM attinet, poterit concludi. Omnium autem h
lutio pendebit a resolutione huius

$$ds + ssdz = Zdz.$$

11. Si ergo proponatur haec aequatio

$$ds + s^2dz = a^2z^{2n}d$$

quae est ea ipsa quam Com. Riccati resolvenda

$$Z = a^2z^{2n} \text{ et } \int dz\sqrt{Z} =$$

atque

$$lZ = 2\,la + 2\,nl$$

Hinc erit

$$t = \frac{2\,abz^{n+1}}{n+1} \text{ et } u = bla$$

Quia autem est

$$t = \frac{2\,abz^{n+1}}{n+1},$$

erit

$$lt = l\frac{2\,ab}{n+1} + (n+1)lz \text{ seu } lz = \frac{lt}{}$$

Quo valore in aequatione altera $u = bla +$
pro lubitu auctis vel diminutis habebitur ist

st aequatio inter t et u, et indicat curvam BN esse logarithmicam, cuiu
gens constans est $= \frac{nb}{n+1}$.

. Pro hoc ergo casu construatur (Fig. 2) logarithmica DN ad asym
AB, cuius subtangens sit $= \frac{nb}{n+1}$. Producatur quaecunque applicat
uae pro axe habeatur, et motu tractorio filum longitudinis b alter

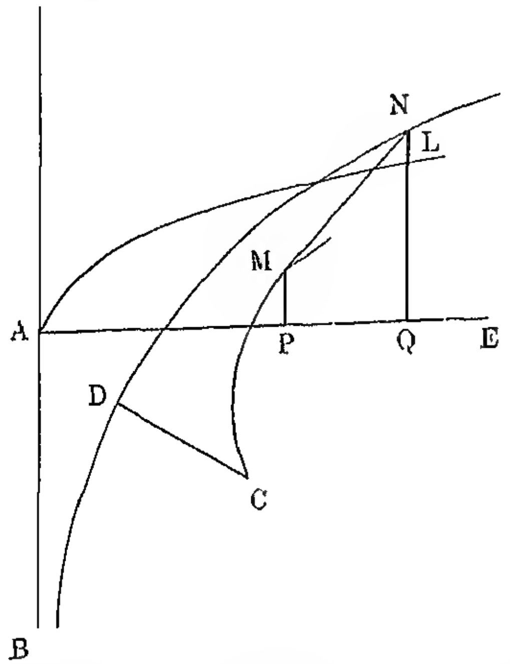

Fig. 2

o in logarithmica protrahatur, describatque alter terminus tractoriam
omittantur ex punctis M et N perpendicula MP et NQ, erit[1])

$$s = \frac{\sqrt{Z}\cdot PQ}{b+QN-PM} = \frac{az^n\cdot PQ}{b+QN-PM}$$

$$z = \sqrt[n+1]{\frac{(n+1)AQ}{2ab}}.$$

Editio princeps:

$$s = \frac{Z\cdot PQ}{2b\sqrt{Z}-PQ} = \frac{a^2z^{2n}\cdot PQ}{2abz^n-PQ} \text{ sumto } z = \sqrt[n+1]{\frac{2(n+1)AQ}{ab}}$$

Correxit H. D.

possunt construi, dummodo sit tangens MN seu filum
logarithmicae ut $n+1$ ad n.

13. Sequente praeterea modo aequatio

$$ds + ssdz = a^2 z^{2n} dz$$

potest construi. Super axe construatur curva paraboloide
$QL = z$, hac aequatione expressa

$$z^{n+1} = \frac{(n+1)t}{2ab}.$$

Deinde filo longitudinis b super logarithmica DN, ut a
describatur tractoria CM. Tum in paraboloide sumatur
eaque producatur, donec logarithmicam secet in N. Ex N
longitudinis b ad tractoriam, et ex M demittatur perpendic
factis erit[1])

$$s = \frac{(n+1)AQ \cdot PQ}{2b \cdot QL(b + QN - PM)}.$$

Vel etiam posita tangente dimidii anguli $MNQ = q$, erit[2])

$$s = \frac{(n+1)AQq}{2b \cdot QL}.$$

14. Cum methodus, qua in reductione aequatio
descriptionem tractoriae sum usus, maximam habeat utili
problematum generalium, quae ad methodum tangentium i
hic nonnulla huiusmodi problemata adiungam eorum
modum ostendam. Cuius rei ratio quo facilius percipiatu
est, quam variis modis natura cuiusque curvae possit dot
sint illi modi, ex quibus facillime diiudicari possit, an
algebraica, an transcendens.

---

1) Editio princeps: $s = \dfrac{4(n+1)^2 AQ^3 \cdot PQ}{bb(4(n+1)AQ \cdot QL - PQQL^2)}$

2) Editio princeps: $s = \dfrac{2(n+1)AQq}{b \cdot QL}$

plicatam, quippe ex qua quaelibet curvae puncta facillime possunt inve huiusmodi aequatione sponte sequitur, utrum curva sit algebraica an se m si aequatio est algebraica, curva quoque talis censetur, sin vero aequ erit transcendens, curva quoque transcendens habetur. Eadem vero sio deduci potest ex aequatione inter alias rectas lineas, quae cur turam exprimat, si modo positio earum rectarum non ab ipsa curva pend l vel ad datum punctum vel datam lineam referatur.

16. At si positio earum linearum, inter quas aequatio curvae natu primit, sine curvae ipsius cognitione definiri non potest, ex ea aequati am singula curvae puncta immediate inveniri non possunt. Ex huiusn oque aequatione, etsi est algebraica, tamen non sequitur curvam esse a aicam, sed saepe maxime erit transcendens. Quamobrem tum ad const nem tum ad cognitionem curvae huiusmodi aequatio in aliam est tr tanda, quae sit inter lineas, quarum positio a curva non pendeat.

17. Optimum igitur ad cognoscendam et construendam curvam re un erit aequationem, si fuerit inter lineas, quarum positio ab ipsa cu ndeat, transmutare in aequationem consuetam inter abscissam et applicat hoc autem negotio summa cura est adhibenda, ne in prolixissimos calc resolutu difficillimas aequationes incidamus. Facillima enim videtur nsmutatio in aequationem inter abscissam et applicatam, sed hoc m erumque in inextricabiles tricas delabimur; id quod unico exemplo ostend fficiet.

18. Exprimatur (Fig. 3) curvae AM natura aequatione inter norma curvam MN et portionem axis AN; quarum MN vocetur u et AN que aequatio curvae naturam exprimens haec simplex admodum $u^2 =$ nunc ponatur abscissa $AP = x$ et applicata $PM = y$, atque cur mentum, quod est $\sqrt{(dx^2 + dy^2)} = ds$, erit

$$MN = u = \frac{yds}{dx} \text{ et } AN = t = x + \frac{ydy}{dx}.$$

nare si hi valores in aequatione substituantur, habebitur quidem h quatio

inter x et y, ex qua neque constructio curvae appareat, neque ...
algebraica an secus.

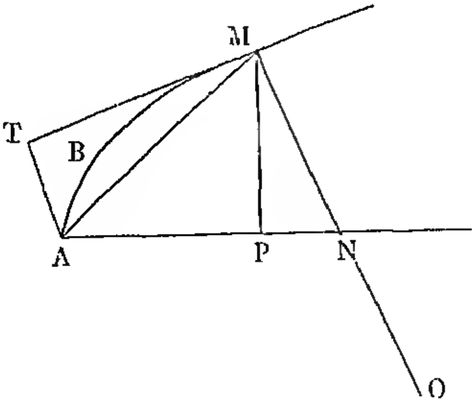

Fig. 3

19. In hoc quidem casu aequatio inventa

$$y^2 ds^2 = ax dx^2 + ay dx dy,$$

quia differentialia duas tantum habent dimensiones, in aequatio...
dimensionis mutari potest, prodibit enim posito $dx^2 + dy^2$ loco ds^2 ...
radice quadrata haec aequatio

$$2 y dy = a dx \pm dx \sqrt{(a^2 + 4 ax - 4 y^2)},$$

ex qua autem non tam facile natura curvae cognoscitur. Ex quo int...
magis compositam aequationem inter t et u assumsissemus, tum ...
ad aequationem differentialem unius dimensionis perveniri potuiss...
tamen a Cel. BERNOULLIO in Act. Lips. ostensum est[1]), quoties det...
algebraica inter t et u, toties quoque aequationem inter x et y fore al...

20. Hanc ob rem alia via est procedendum, si ex aequatione ...
aequationem inter x et y eruere velimus, atque hoc observavi com...
effici posse eadem methodo, qua ante constructionem aequationis

$$ds + ssdz = Zdz$$

ad motum tractorium reduxi. Hac enim methodo statim appare...

---

1) Vide IOH. BERNOULLII *Lectiones mathematicae de methodo integralium aliisque ... usum Ill. Marchionis Hospitalii*, Lectio 13. *Opera omnia*, t. III, p. 431.

essitatem, a qua curvae constructio pendet.

. Retineamus igitur eundem casum sitque aequatio inter $AN = t$ e
$= u$ quaecunque; maneant etiam

$$AP = x,\ PM = y \text{ et } \sqrt{(dx^2 + dy^2)} = ds,$$

$$t = x + \frac{ydy}{dx} \text{ et } u = \frac{yds}{dx}.$$

ır $dy = pdx$; erit

$$t = x + py \text{ et } u = y\sqrt{(1 + pp)} \text{ seu } y = \frac{u}{\sqrt{(1 + pp)}}.$$

entietur haec aequatio, habebitur

$$dy = pdx = \frac{du}{\sqrt{(1 + pp)}} - \frac{updp}{(1 + pp)^{\frac{3}{2}}},$$

quatio autem differentiata dat

$$dt = dx + ppdx + ydp$$

pdx loco dy, ex qua obtinetur

$$dx = \frac{dt}{1 + pp} - \frac{ydp}{1 + pp};$$

er p multiplicata locoque y eius valore substituto dat

$$pdx = \frac{pdt}{1 + pp} - \frac{pudp}{(1 + pp)^{\frac{3}{2}}};$$

m illa coniuncta prodit

$$\frac{pdt}{\sqrt{(1 + pp)}} = du.$$

2. Ex hac aequatione inventa statim obtinetur[1])

$$p = \frac{du}{\sqrt{(dt^2 - du^2)}} \text{ et } \sqrt{(1 + pp)} = \frac{dt}{\sqrt{(dt^2 - du^2)}}.$$

Ponendo $dt = 0$ seu $t = u =$ constanti, prodibunt circuli. H. D.

$dt \qquad dt$

Quamobrem si aequatio inter t et u fuerit algebraica, aequatio in
quoque erit algebraica, ex eaque constructio curvae quaesitae facile fl
a qua quadratura pendet aequatio inter t et u, ab eadem quadratura
aequatio inter x et y, et consequenter quoque constructio ipsius curv

23. In casu speciali, quem ante considerabamus, erat $u^2 = a$

$$t = \frac{u^2}{a} \text{ et } dt = \frac{2\,udu}{a}$$

atque

$$\sqrt{(dt^2 - du^2)} = \frac{du}{a}\sqrt{(4\,u^2 - a^2)}.$$

His igitur substitutis proveniet

$$y = \frac{1}{2}\sqrt{(4\,u^2 - a^2)} \text{ atque } x = \frac{u^2}{a} - \frac{a}{2}.$$

Haec autem dat

$$4\,u^2 = 4\,ax + 2\,a^2;$$

qui ipsius $4\,u^2$ valor in illa aequatione substitutus dat hanc inter x e
tionem algebraicam

$$2\,y = \sqrt{(4\,ax + aa)} \text{ hoc est } y^2 = ax + \frac{a^2}{4},$$

quae est aequatio pro parabola abscissis in axe ex foco sumtis.

24. Si (Fig. 4) curvae AM tangens MT ad axem PA usque p
atque ex A ad axem perpendicularis AV erigatur, detur aequatio in
AV, qua curvae natura exprimitur; oporteatque invenire aequatio
abscissam AP et applicatam PM, seu construere curvam, quae om
per puncta T et V ductas tangat. Positis

$$AT = t,\; AV = u \text{ et } AP = x,\; PM = y$$

erit

e ponitur relatio inter t et u, quae sit quaecunque.

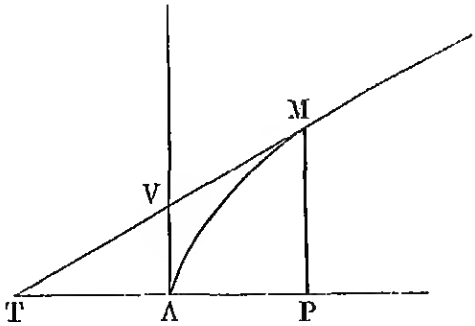

Fig. 4

Sit nunc $dy = p\,dx$, erit

$$t = \frac{y}{p} - x \text{ et } u = y - px.$$

ero posterior aequatio differentiata posito $p\,dx$ loco dy dat

$$du = -x\,dp \text{ et } x = \frac{-du}{dp}.$$

res vero in priore aequatione loco x et y substituti dant $t = \frac{u}{p}$ seu
Erit ergo

$$dp = \frac{t\,du - u\,dt}{tt}$$

$$x = \frac{tt\,du}{u\,dt - t\,du} \text{ et } y = u + \frac{ut\,du}{u\,dt - t\,du}.$$

erum patet, quoties aequatio inter t et u fuerit algebraica, toties curvam oque fore algebraicam, propter aequationem inter x et y algebraicam

Manente aequatione inter AT, t et AV, u quaecunque, si loco rec super axe AT verticibus T infinitae parabolae TVM describantu ncta V transeuntes, invenienda proponitur curva AM, quae ab hi is omnibus tangatur. Positis

$$AP = x \text{ et } PM = y \text{ et } dy = p\,dx,$$

Quia porro parabola TVM tangere debet curvam AM,
tangentem in puncto M atque ideoquoque subtangenter
subtangens parabolae in $M = 2PT = 2t + 2x$, qua

$$\frac{ydx}{dy} = \frac{y}{p}$$

subtangenti curvae quaesitae AM, unde oritur

$$y = 2pt + 2px.$$

27. Harum duarum aequationum si prior per
prodit $y = \frac{u^2}{2pt}$, quo valore in altera aequatione substit

$$x = \frac{u^2}{4p^2t} - t.$$

Differentietur nunc utraque aequatio; erit

$$dy = pdx = \frac{udu}{pt} - \frac{u^2dt}{2ptt} - \frac{u^2dp}{2p^2t}$$

et

$$dx = \frac{udu}{2p^2t} - \frac{u^2dt}{4p^2tt} - \frac{u^2dp}{2p^3t} - dt.$$

Ex quibus aequationibus dx eliminato prodit

$$\frac{udu}{2pt} + pdt = \frac{u^2dt}{4pt^2} \text{ seu } pp = \frac{u^2}{4tt} -$$

Hinc ergo erit

$$x = \frac{2ttdu}{udt - 2tdu} \text{ et } y = \frac{u^2\sqrt{dt}}{\sqrt{(u^2dt - 2tu}}$$

Ex quo perspicitur curvam AM toties esse algebra
inter t et u talis fuerit.

28. Duo haec posteriora problemata alio quide
possunt quaerendo punctum, quo duae curvae proxima
erit contactus curvae quaesitae AM. Semper autem

...io ad aequationem algebraicam inter x et y per plures differentiatiiones perveniri queat.

... Si (Fig. 5) infinitae rectae RN intra angulum rectum A quomo...que fuerint dispositae, ita ut earum positio exprimatur aequation...

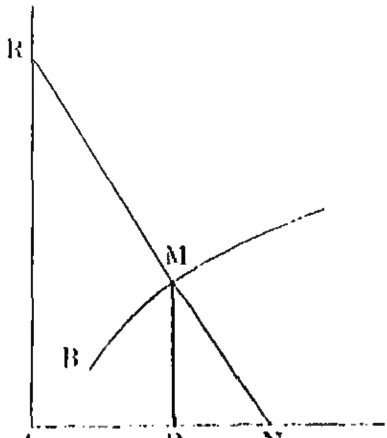

Fig. 5

...que inter AN, t et AR, u, invenienda proponatur curva BM, qua... has rectas ad angulos rectos traiiciat. Positis

$$AP = x,\ PM = y \text{ et } dy = p\,dx$$

$$PN = \frac{y\,dy}{dx} = py,$$

N in curvam est normalis; ideoque $t = x + py$; deinde est

$$dy : dx = p : 1 = t : u,$$

...erit

$$t = pu \text{ seu } p = \frac{t}{u} \text{ et } dy = \frac{t\,dx}{u}.$$

...am vero aequationem est

$$y = u - \frac{ux}{t};$$

in qua aequatione duae insunt variabiles x et t, quia u per t

30. Aequatio postrema reducta in hanc abit

$$dx + x\left(\frac{tdt + udu}{tt + uu} - \frac{dt}{t}\right) = \frac{tudu}{tt + uu},$$

quae per $\frac{\sqrt{(tt + uu)}}{t}$ multiplicata fit integrabilis; erit autem

$$x = \frac{t}{\sqrt{(tt + uu)}}\int\frac{udu}{\sqrt{(tt + uu)}};$$

quo cognito habebitur simul

$$y = u - \frac{u}{\sqrt{(tt + uu)}}\int\frac{udu}{\sqrt{(tt + uu)}}.$$

Quoties ergo

$$\frac{udu}{\sqrt{(tt + uu)}}$$

integrationem admittit, toties curva BM erit algebraica[1]
constructio pendet a quadratura

$$\int\frac{udu}{\sqrt{(tt + uu)}}.$$

31. Consideremus huius problematis casum, quo RN
magnitudinis manet; seu quo

$$\sqrt{(tt + uu)} = a, \text{ vel } u = \sqrt{(a^2 - t^2)}.$$

Erit ergo

$$\int\frac{udu}{\sqrt{(tt + uu)}} = \frac{-tt}{2a};$$

ubi constantem non adiicio, ne ad maxime compositas ae
Hoc invento erit

$$x = \frac{-t^3}{2a^2} \text{ atque } t = -\sqrt[3]{2a^2x},$$

1) Dummodo integrale sit algebraicum.

ero est

$$y = \frac{u(t-x)}{t},$$

$$y = \frac{-(x+\sqrt[3]{2a^2x})}{\sqrt[3]{2a^2x}}\sqrt{(a^2-\sqrt[3]{4a^4x^2})},$$

mendis quadratis transit in hanc

$$\frac{3ax^2}{\sqrt[3]{4ax^2}} = a^2 - x^2 - y^2,$$

mendis cubis in sequentem:

$$(a^2 - x^2 - y^2)^3 = \frac{27}{4}a^2x^4,$$

t pro linea sexti ordinis.

Dantur autem praeter hanc curvam infinitae aliae quaestioni aequ ientes, quae invenientur, si ad integrale ipsius $\frac{udu}{\sqrt{(tt+uu)}}$ quantita nque constans addatur. Maxime autem aequatio inter x et y erit com propterea quod ex aequationibus indeterminata t eliminari debet, qua nor dimensiones ascendit. Interim tamen constructio erit facilis.

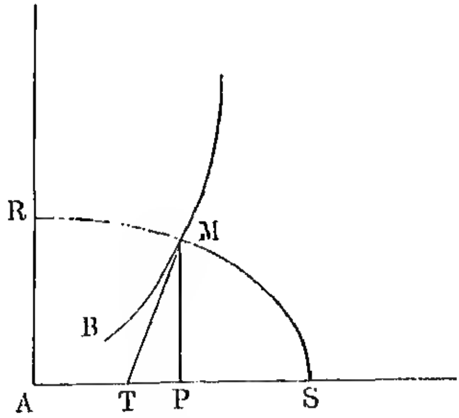

Fig. 6

Simili modo problema solvi potest, si loco rectarum puncta R et tium curvae quaecunque per haec puncta ducantur, quae a quaesita a

Infinitas vero has ellipses ad angulos rectos traiiciat curva BM, q
Ponantur

$$AP = x \text{ et } PM = y \text{ atque } dy = pdx,$$

erit ex natura ellipsis

$$y = \frac{u}{t}\sqrt{(tt - xx)}, \text{ seu } y^2 = u^2 - \frac{u^2x^2}{t^2}.$$

34. Ad ellipsin in puncto M ducatur normalis MT; erit
ditionem problematis simul tangens curvae quaesitae BM. Qu
MT est normalis in ellipsin, erit

$$PT = \frac{u^2x}{t^2}.$$

At quatenus MT est tangens curvae BM, erit

$$PT = \frac{ydx}{dy} = \frac{y}{p}.$$

Quocirca habebitur ista aequatio

$$y = \frac{pu^2x}{t^2};$$

cuius differentialis est

$$dy = pdx = \frac{pu^2dx}{t^2} + \frac{2puxdu}{tt} + \frac{u^2xdp}{tt} - \frac{2pu^2xdt}{t^3}$$

ex qua fit

$$pdx = \frac{2ptuxdu + tu^2xdp - 2pu^2xdt}{t(t^2 - u^2)}.$$

Prior vero aequatio differentiata dat

$$ydy = \frac{p^2u^2xdx}{tt} = udu - \frac{u^2xdx}{t^2} - \frac{ux^2du}{tt} + \frac{u^2x^2dt}{t^3}$$

seu

$$uxdx = \frac{t^3du - tx^2du + ux^2dt}{t(pp + 1)}.$$

$$u^2x^2 = \frac{\ldots}{(pp+1)(2\,ptdu + tudp - 2\,pudt)}.$$

...les vero aequationes coniunctae y eliminata dant

$$x^2 = \frac{t^4}{tt + ppuu}.$$

...ius x^2 valor si in illa aequatione substituatur, proveniet

$$(pp+1)\ (2\,ptdu + tudp - 2\,pudt) = p(tt - uu)\,(p^2udu + tdt).$$

...r

$$p = \frac{qtt}{uu},$$

... ista aequatio

$$\frac{tudq}{q} = \frac{(tt - uu)(q^2t^3du + u^3dt)}{q^2t^4 + u^4},$$

...s aequationis constructione vel separatione ipsius q ab u et t pende...
...ctio curvae quaesitae.

... Habeat exempli causa AR ad AS rationem datam, seu sint omne...
... inter se similes, erit $u = nt$; atque generalis aequatio abibit in han...

$$\frac{dq}{q} = \frac{(1 - nn)(q^2dt + n^2dt)}{q^2t + n^4t},$$

... variabiles t et q separari possunt, prodibit namque

$$\frac{(1 - nn)dt}{t} = \frac{(q^2 + n^4)dq}{q(q^2 + n^2)} = \frac{n^2dq}{q} + \frac{(1 - n^2)qdq}{q^2 + n^2},$$

...ntegrata dat

$$\left(\frac{t}{\sqrt{(q^2 + n^2)}}\right)^{1-nn} = Cq^{n^2} \text{ seu } t = aq^{\frac{n^2}{1-n^2}}\sqrt{(q^2 + n^2)}.$$

...rgo

$$u = naq^{\frac{n^2}{1-n^2}}\sqrt{(q^2 + n^2)} \text{ et } x = naq^{\frac{n^2}{1-n^2}} \text{ et } y = qx.$$

... et y ergo elicitur ista aequatio

$$x = b^{1-n^2}\,y^{n^2}$$

pro curvis parabolicis; quod congruit cum iis, quae de ...
nalibus iam pridem sunt detecta.

37. Quando in astronomia physica ex data vi centri...
minatur, quam corpus proiectum describit, pervenitur stati...
inter distantiam corporis a centro virium et perpendicul...
tangentem curvae demissum. Difficulter autem ex tali aeq...
potest, utrum curva descripta sit algebraica an transcenden...
est aequationem inter coordinatas orthogonales simplici...
Methodo vero nostra hactenus usitata haec quaestio facile ex...

38. Sit (Fig. 3) centrum virium A et curva a corpore ...
BM[1]); ponatur distantia $AM = t$ et in tangentem MT ex...
pendiculum $AT = u$, sitque curvae natura aequatione inte...
In axe per A pro libitu ducto sit

$$\text{abscissa } AP = x, \text{ applicata } PM = y, \text{ et } dy = \ldots$$

erit

$$t = \sqrt{(x^2 + y^2)} \text{ et } u = \frac{y - px}{\sqrt{(1 + pp)}}.$$

Haec posterior aequatio vero differentiata dat

$$du\sqrt{(1 + pp)} + \frac{updp}{\sqrt{(1 + pp)}} = -xdp,$$

unde erit

$$x = \frac{-du\sqrt{(1 + pp)}}{dp} - \frac{pu}{\sqrt{(1 + pp)}} \text{ et } y = \frac{-pdu\sqrt{(1 + pp)}}{dp} \ldots$$

39. Substituantur hi ipsorum x et y valores in aequat...
quo facto habebitur

$$tt = u^2 + \frac{du^2(1 + pp)^2}{dp^2},$$

unde oritur

$$\frac{dp}{1 + pp} + \frac{du}{\sqrt{(tt - uu)}} = 0.$$

Denotet

$$\int \frac{du}{\sqrt{(tt - uu)}}$$

---

1) A non solet esse curvae punctum.

$$dp^2 + dq^2 = \ldots$$

...nte A arcum cuius tangens est quantitas adiuncta. Quocirca erit

$$p = \frac{b-q}{1+bq}, \text{ et } \sqrt{(1+pp)} = \frac{\sqrt{(1+bb)(1+qq)}}{1+bq}.$$

...utem sit

$$\frac{du}{dp} = \frac{-\sqrt{(tt-uu)}}{1+pp} = \frac{-(1+bq)^2\sqrt{(tt-uu)}}{(1+bb)(1+qq)},$$

$$x = \frac{(1+bq)\sqrt{(tt-uu)} - (b-q)u}{\sqrt{(1+bb)(1+qq)}}$$

$$y = \frac{(b-q)\sqrt{(tt-uu)} + (1+bq)u}{\sqrt{(1+bb)(1+qq)}}.$$

... Quoties ergo aequatio inter t et u est algebraica simulque ita com... ut $\int \frac{du}{\sqrt{(tt-uu)}}$ denotet arcum circuli, cuius tangens algebraice potes...ri, toties curva a corpore descripta erit algebraica, eiusque aequati... coordinatas orthogonales algebraica per inventas formulas invenitur.

... Si detur relatio inter radium osculi MO et partem eius MN se...em aequatione quacunque, aequatio inter coordinatas AP, PM ha... potcrit inveniri, ex qua statim appareat quibus casibus curva fi...aica. Sit nempe $MN = t$ et $MO = u$ dataque sit aequatio quaecunqu... et u; ponatur

$$AP = x, \; PM = y \text{ atque } dy = pdx.$$

...rgo elementum curvae

$$= dx\sqrt{(1+p^2)} \text{ et } ddy = dpdx$$

...dx constante. Ex his igitur erit

$$MN = t = y\sqrt{(1+pp)} \text{ et } MO = u = \frac{-dx(1+pp)^{\frac{3}{2}}}{dp}.$$

prior differentiata dat

$$dy = pdx = \frac{dt + ppdt - ptdp}{(1+pp)^{\frac{3}{2}}}.$$

His ergo aequationibus coniunctis habebitur

$$pudp = ptdp - dt - ppdt.$$

42. Aequatio haec inventa, quia u per t dari ponit
bilium separationem, abit enim in hanc

$$\frac{pdp}{1+pp} = \frac{dt}{t-u},$$

cuius integralis est

$$l\sqrt{(1+pp)} = \int \frac{dt}{t-u}.$$

Sit

$$\int \frac{dt}{t-u} = lq,$$

erit

$$\sqrt{(1+pp)} = q \text{ et } y = \frac{t}{q}.$$

Hinc ergo porro est

$$dy = \frac{qdt - tdq}{qq} = pdx = dx\sqrt{(qq-1)};$$

ideoque

$$x = \int \frac{qdt - tdq}{qq\sqrt{(qq-1)}}.$$

Ex quo perspicitur, ut curva AM fiat algebraica, duo requi

$$\int \frac{dt}{t-u}$$

logarithmis possit exhiberi, atque tum, ut

$$\frac{qdt - tdq}{qq\sqrt{(qq-1)}}$$

integrationem admittat[1]).

1) Necesse est insuper integrale algebraice exprimi posse. Quod non fi
utar, $u = -t$, $t = aq^2$. Cf. notam p. 98.

$$q = a^{m-1} t^{1-m} \text{ atque } y = \frac{t^m}{a^{m-1}}.$$

autem porro

$$dy = \frac{mt^{m-1}dt}{a^{m-1}} = p\,dx = dx\sqrt{(a^{2m-2}\,t^{2-2m} - 1)},$$

c fit

$$dx = \frac{mt^{2m-2}dt}{a^{m-1}\sqrt{(a^{2m-2} - t^{2m-2})}} \text{ atque } x = \int \frac{mt^{2m-2}dt}{a^{m-1}\sqrt{(a^{2m-2} - t^{2m-2})}}.$$

quo perspicitur curvam fore algebraicam, si haec formula fuerit integrabi autem evenit, quoties vel $\frac{m}{m-1}$ fuerit numerus impar affirmativus $\frac{2i+1}{2i}$, vel $\frac{m}{1-m}$ numerus par affirmativus seu[2]) $m = \frac{2i}{2i+1}$ denota merum integrum affirmativum[3]). Casus autem quo $n = 1$ dat $t = u$ a $= 0$ seu $t = u =$ constanti, ex quo cognoscitur curvam esse circulum.

44. Data sit nunc aequatio quaecunque inter arcum AM et rad uli MO, ex qua determinari debeat aequatio inter coordinatas AP et od antequam quomodo inveniendum sit ostendam, observari conv c curvas exprimendi rationem per aequationem inter arcum et rad uli maxime ad curvas cognoscendas esse accomodatam. Aequatio r coordinatas orthogonales, vel inter radium et perpendiculum gentem tam varias et diversas formas sumendis aliis axibus alii issarum initiis induere potest, ut, ad quamnam curvam perti mvis curva sit notissima, saepe difficulter perspici possit. Aequ o, quae inter curvam et radium osculi exhibetur pro diversis tan

1) Editio princeps: $u = mt$. Correxit H

2) Editio princeps: $m = \frac{2i+1}{2i+2}$. Si $m = \frac{2i+1}{2i+2}$ formula est integrabilis, sed non est n. H

3) Formula erit integrabilis quoties vel fuerit $m = \frac{2i+1}{i}$, vel $m = \frac{2i}{2i+1}$, denotante m integrum sive positivum sive negativum; sed integrale est algebraicum ea conditione u grum positivum. Cf. notam p. 44. H

at utrum curva esset algebraica an transcendens non tam facile
vero incommodo sequenti modo occurreretur.

45. Sit arcus $AM = s$ et radius osculi $MO = r$ dataq
quaecunque inter s et r. Ponantur $AP = x$, $PM = y$ sitq
hisque positis erit

$$ds = dx\sqrt{(pp+1)} \text{ et } r = \frac{-dx(1+pp)^{\frac{3}{2}}}{dp}.$$

Ex illa vero aequatione est

$$dx = \frac{ds}{\sqrt{(pp+1)}},$$

ex hac autem

$$dx = \frac{-rdp}{(pp+1)^{\frac{3}{2}}}.$$

Quamobrem proveniet haec aequatio

$$ds(pp+1) = -rdp \text{ seu } \frac{-ds}{r} = \frac{dp}{1+pp}.$$

Denotet $\int\frac{ds}{r}$ arcum circuli cuius tangens sit q posito radio $= 1$

$$At\cdot b - At\cdot q = At\cdot p;$$

unde fit

$$p = \frac{b-q}{1+bq} \text{ et } \sqrt{(pp+1)} = \frac{\sqrt{(1+bb)(1+qq)}}{1+bq}.$$

Ex his oritur

$$dx = \frac{(1+bq)\,ds}{\sqrt{(1+bb)(1+qq)}} \text{ et } dy = \frac{(b-q)\,ds}{\sqrt{(1+bb)(1+q}$$

Unde intelligitur, si primo $\int\frac{ds}{r}$ denotet arcum circuli, cuius ta
per q possit exhiberi, atque deinde

1) $At.b$ denotante arcum cuius tangens est b.

ationem admittat, fore curvam algebraicam.

6. Sin autem $\frac{ds}{r}$ absolute potest integrari, fieri quoque potest, ut curv ebraica: ut sit

$$\int \frac{ds}{r} = v,$$

$$A l \cdot p = b - v \text{ et } p = l \cdot A\,(b - v).$$

s fit[1])

$$x = \int ds \cos. A\,(b - v) \text{ et } y = \int ds \sin. A\,(b - v)$$

es ergo haec integralia ita possunt exhiberi, ut nonnisi sin. $A\,(b - v)$ c $(b - v)$ contineant, toties ob[2])

$$1 = \square \sin. A\,(b - v) + \square \cos. A\,(b - v)$$

tio algebraica inter x et y obtinetur. Ut si fuerit $r = a$, erit

$$x = -a \sin. A\,(b - v) \text{ et } y = a \cos. A\,(b - v)$$

ie

$$\frac{x^2}{a^2} + \frac{y^2}{a^2} = 1 \text{ seu } y^2 = a^2 - x^2,$$

tio pro circulo cuius radius est $= a$.

---

cos. $A\,(b - v)$ et sin. $A\,(b - v)$ idem significant quod $\cos\,(b - v)$ et $\sin\,(b - v)$. H. D.

$\square$ sin. $A\,(b - v)$ et $\square$ cos. $A\,(b - v)$ idem significant quod $\sin^2\,(b - v)$ et $\cos^2\,(b - v)$. H. D.

DE INTEGRATIONE AEQUATI
DIFFERENTIALIUM ALTIORUM G

Commentatio 62 indicis Enestroemiani
Miscellanea Berolinensia 7, 1743, p. 193—242

1. Quanquam ad resolvendas aequationes differen plurimae adhuc excogitatae sunt methodi, atque in hoc metrae operam ac studium collocaverunt: tamen parum attulerunt ad aequationes differentiales altiorum graduum vel construendas vel integrandas. Aequationes quidem d gradus ita resolvi solent, ut per idoneam substitutionem a gradus reducantur, quo facto earum resolutio ad viam ma tam revocatur: atque in hoc negotio nonnulla subsidia an excogitavi[1]), quorum ope innumerabiles aequationes di gradus ad primum gradum deprimi, atque adeo sive con possunt. At vero in aequationibus differentialibus terti similia artificia, quibus eae ad gradum inferiorem traduci plerumque nihil prosunt, cum hoc pacto ad aequationes di vel etiam altioris gradus tam complicatas perveniatur, omnino nequeant. Quamobrem in hoc negotio non paru methodus, quam hic sum expositurus, cuius beneficio plu differentiales altiorum graduum sine praevia reductione a statim integrari, atque aequationes integrales in termin possunt.

1) Confer Commentationem 10 huius voluminis.

2. Sint y et x variabiles, quibus aequatio differentialis cuiuscu
dus contineatur, in qua differentiale dx assumtum sit constans, e
peat altera variabilis y cum suis differentialibus dy, ddy, d^3y etc. in sin
minis unicam dimensionem, ita ut aequatio, cuiuscunque demum sit gr
uentem induat formam:

$$0 = Ay + \frac{Bdy}{dx} + \frac{Cddy}{dx^2} + \frac{Dd^3y}{dx^3} + \frac{Ed^4y}{dx^4} + \frac{Fd^5y}{dx^5} + \text{etc.},$$

qua litterae A, B, C, D etc. significent quantitates vel constantes
eram variabilem x utcunque involventes. Manifestum autem est, hanc ae
nem latissime patere, non solum enim ob coefficientes indetermin
B, C, D etc., quos simul functiones quascunque ipsius x assumimus, ma
generalis, sed etiam aequationes differentiales cuiuscunque gradus
nplectitur. Hanc igitur aequationem, quibus casibus integrationem
stat, in hac dissertatione evolvam.

3. Primum quidem perspicuum est, aequationem integralem compl
eter quantitates constantes in ipsa aequatione differentiali contenta
vas constantes arbitrarias in se complecti oportere, quoti fuerit g
quatio differentialis proposita. Quodsi enim ponamus eam aequati
e differentialem gradus n, ita ut ultimus illius terminus sit

$$\frac{Nd^ny}{dx^n},$$

unam integrationem ea reducetur ad gradum $n-1$, per duas int
nes successive institutas ad gradum $n-2$, per tres ad gradum $n-3$ e
rro. Ex quo intelligitur, demum post n integrationes ad aequationem
lem terminis finitis expressam perveniri. Quoniam vero per unamquan
egrationem una constans arbitraria in integrale ingreditur, manifestum
egrale completum n constantes arbitrarias complecti oportere.

4. Aequatio igitur integralis completa tot constantes arbitrarias
nplectitur, quot exponens n continebit unitates; haecque aequatio integ
que late patere censenda est, atque ipsa aequatio differentialis gradus n:
llus valor finitus pro y assumtus aequationi differentiali satisfacere quea
n contineatur in aequatione integrali completa. Quodsi autem in ista ae
ne integrali completa una pluresve illarum constantium arbitrariarur

quae non omnes possibiles valores ipsius y ...
in se complectitur. Probe igitur discerni oportet aequat...
completam a particulari; atque si aequationi differentiali ...
velimus, aequationem integralem completam inveniri oport...

5. Ad cognoscendum autem, utrum aequatio integ...
completa, nec ne, criterium ex allatis facile colligitur. Prim...
tioni propositae differentiali satisfacere debet, quod fit, d...
tione aequatio identica resultat; alioquin enim illa aequat...
integralis. Praeterea vero necesse est, ut aequatio integ...
quantitates constantes arbitrarias, quoti fuerit gradus aeq...
proposita. Si enim pauciores in ea insint constantes, tum ...
data non erit completa, sed tantum particularis. In enumer...
stantium arbitrariarum probe cavendum est, ne per num...
litterarum fallamur, neque pro diversis quantitatibus habe...
invicem determinantur.

6. Quo discrimen inter aequationes integrales comple...
clarius intelligatur, iuvabit rem exemplo illustrasse. Sit igi...
aequatio differentialis

$$aady + yydx = (aa + xx)\,dx;$$

cui satisfacere patet hunc valorem $y = x$, quippo qui su...
aequationem identicam. Est igitur $y = x$ aequatio integrali...
pleta, cum ea neque constantem a, quae in aequatione differ...
praeterea aliam constantem arbitrariam contineat, quema...
differentialis primi gradus postulat. Vehementer igitur fa...
aequationem $y = x$ pro integrali completa huius

$$aady + yydx = (aa + xx)\,dx$$

venditare vellet; aequatio enim integralis completa est

$$y = x + \frac{aabe^{\frac{-xx}{aa}}}{aa + b\int e^{\frac{-xx}{aa}}dx};$$

7. Simili modo videmus huic aequationi differentio-differentiali

$$y = \frac{xdy}{dx} + \frac{axddy}{dx^2}$$

facere hanc aequationem finitam $y = x$; procul autem abest, quon integralis completa omnemque vim aequationis differentio-differen uriat, quoniam aequatio integralis completa praeter constantem a ntitates arbitrarias continere debet. Videmus vero etiam hanc aequati nx satisfacere, quae autem, quia unicam constantem n continet, tan o est particularis. Aequatio autem integralis completa est

$$y = nx + bx\int\frac{e^{\frac{-x}{a}}dx}{xx},$$

praeter constantem a duas continet constantes arbitrarias b et n ra rei postulat.

8. Cum autem omnes aequationes integrales particulares in com inentur, patet ex pluribus integralibus particularibus completam e; atque adeo ex integralibus particularibus integrale completum tur. Saepenumero quidem aeque difficile est ex cognitis aliquot inte particularibus integrale completum vel saltem integrale latius pa ne idem ex ipsa aequatione differentiali per integrationem colliger aequatio, quam tractare suscepimus[1]),

$$0 = Ay + \frac{Bdy}{dx} + \frac{Cddy}{dx^2} + \frac{Dd^3y}{dx^3} + \frac{Ed^4y}{dx^4} + \text{etc.}$$

est comparata, ut cognitis valoribus particularibus ipsius y duobus ve ex iis facile valor latius patens illos nempe valores in se comple ns y formari queat. Hocque pacto ex sufficienti numero valorum pa um pro y inventorum valor completus, seu aequatio integralis com cinnari poterit.

1) Vide epistulam ab Eulero ad I. Bernoulli 15. 9. 1739 scriptam (n. 863 indicis En), *Bibl. math.* 63, 1905, 37/38. Vide quoque Commentationem 188 huius voluminis et *n calculi integralis* vol. II, § 775--778, 842—846, 1117--1137. *Leonhardi Euleri Opera* I, vol. 20 et 12.

$$Ay + \frac{Bdy}{dx} + \frac{Cddy}{dx^2} + \text{etc.} = 0,$$

tum valor αp loco y substitutus eandem expressionem e
hocque modo una constans arbitraria α in aequationem
larem $y = p$ introduci potest. Sin autem praeterea l
satisfaciat propositae, tum pari modo quoque satisfaciet y
duobus valoribus particularibus $y = \alpha p$ et $y = \beta q$
patens

$$y = \alpha p + \beta q.$$

Si enim expressio

$$Ay + \frac{Bdy}{dx} + \frac{Cddy}{dx^2} + \frac{Dd^3y}{dx^3} + \text{etc.}$$

nihilo aequalis redditur, posito tam αp quam βq loco
eandem expressionem nihilo aequalem fieri debere, si loco

10. Simili modo si p, q, r, s etc. fuerint eiusmodi f
singulae seorsim loco y substitutae expressionem

$$Ay + \frac{Bdy}{dx} + \frac{Cddy}{dx^2} + \text{etc.}$$

evanescentem efficiant, tum etiam hic valor

$$\alpha p + \beta q + \gamma r + \delta s + \text{etc.}$$

loco y substitutus eandem expressionem nihilo aequale
p, q, r, s etc. fuerint valores particulares ipsius y, qui ipsi
sita conveniunt, tum ex iis colligitur iste valor longe lat

$$y = \alpha p + \beta q + \gamma r + \delta s + \text{etc.}$$

aequationi propositae pariter satisfaciens. Hicque valo
si tot affuerint constantes arbitrariae $\alpha, \beta, \gamma, \delta$ etc., quoti
differentialis proposita. Facilem igitur nacti sumus
valoribus particularibus ipsius y eius valorem compl
omnes omnino valores ipsius y aequationi satisfaciente
sicque habebitur aequatio integralis in terminis finitis c

is propositae

$$0 = Ay + \frac{Bdy}{dx} + \frac{Cddy}{dx^2} + \frac{Dd^3y}{dx^3} + \ldots + \frac{Nd^ny}{dx^n}$$

reducitur, ut valores particulares investigemus, qui pro y subs[tituti ae]quationem identicam reddant. Tot autem eiusmodi valoribus particula[ribus] opus, quoad iis praescripto modo colligendis tot constantes arbitr[ariae ...]erint, quot exponens maximus n continet unitates. Quare si sin[gulae ae]quationes particulares unam secum gerant constantem arbitrariam, eius[modi ae]quationes numero n requiruntur ad aequationem integralem compl[etam con]stituendam. Sin autem quaedam harum aequationum particularium p[lures] constantes arbitrarias implicent, tum eo paucioribus opus erit aequatio[nibus par]ticularibus ad completam ex iis colligendam.

12. Denotent iam omnes litterae A, B, C, D etc. quantitates consta[ntes ...] ut integrari debeat haec aequatio differentialis gradus n

$$0 = Ay + \frac{Bdy}{dx} + \frac{Cddy}{dx^2} + \frac{Dd^3y}{dx^3} + \ldots + \frac{Nd^ny}{dx^n}.$$

[Qu]oniam y cum suis differentialibus ubique unicam dimensionem con[stituit, se]cundum methodum meam in Tomo III. Commentariorum Academiae [...]tanae[1]) traditam haec aequatio differentialis uno gradu deprimetu[r. Pon]amus

$$y = e^{\int p\,dx},$$

[er]o singula differentialia ipsius y erunt

$$\frac{dy}{dx} = e^{\int p\,dx}p$$

$$\frac{ddy}{dx^2} = e^{\int p\,dx}\left(pp + \frac{dp}{dx}\right)$$

$$\frac{d^3y}{dx^3} = e^{\int p\,dx}\left(p^3 + \frac{3\,pdp}{dx} + \frac{ddp}{dx^2}\right)$$

$$\frac{d^4y}{dx^4} = e^{\int p\,dx}\left(p^4 + \frac{6\,ppdp}{dx} + \frac{4\,pddp}{dx^2} + \frac{3\,dp^2}{dx^2} + \frac{d^3p}{dx^3}\right)$$

etc.,

[qui] valores si in proposita substituantur, ea dividi poterit per $e^{\int p\,dx}$, [et re]manebit aequatio differentialis gradus $n-1$.

1) Vide p. 13 huius voluminis.

orietur sequens aequatio algebraica:

$$0 = A + Bp + Cp^2 + Dp^3 + Ep^4 + \ldots .$$

ex qua si valor aliquis pro p eruatur, simul habebit
particularis $y = e^{px}$, aequationi differentiali propositae
ergo etiam uti vidimus haec aequatio $y = a e^{px}$, quo
constans ac radix huius aequationis algebraicae

$$0 = A + Bp + Cp^2 + Dp^3 + \ldots +$$

14. Perduximus ergo inventionem valorum par
bili y ad resolutionem aequationis algebraicae n dim
mamus hanc

$$0 = A + Bz + Cz^2 + Dz^3 + \ldots +$$

huiusque aequationis singulae radices seu divisores d
particulares ipsius y. Si enim fuerit $pz - q$ divisor isti
oritur $z = \frac{q}{p}$, erit

$$y = a e^{\frac{qx}{p}};$$

qui valor particularis unam continet constantem arbi
illa aequatio algebraica n dimensionum contineat n ra
quoque orientur n valores particulares pro y; qui
valorem universalem pro y; hicque simul erit valor
contineat constantes arbitrarias; quod est criterium
completae.

15. Si ergo aequationis istius algebraicae n dime
fuerint reales, tum prodibit valor completus pro y i
pressus, eritque aggregatum n formularum expone
$a e^{qx:p}$, hocque adeo casu integrale completum per so
quadraturam hyperbolae exprimi poterit. Quodsi a
illius aequationis algebraicae fuerint imaginariae, tum

esve radices aequationis sint inter se aequales; tum enim ob p
nulas exponentiales aequales numerus constantium arbitrariarum
tur atque ob hanc causam integrale inventum non amplius erit compl

16. Utrique incommodo medelam afferemus, si nexum inter a
nem differentialem propositam

$$0 = Ay + \frac{Bdy}{dx} + \frac{Cddy}{dx^2} + \frac{Dd^3y}{dx^3} + \dots\dots + \frac{Nd^ny}{dx^n}$$

ue inter aequationem algebraicam formatam

$$0 = A + Bz + Cz^2 + Dz^3 + \dots\dots + Nz^n$$

ntius contemplemur. Quemadmodum enim ex hac illa oritur, si lo
atur y, loco z vero $\frac{dy}{dx}$, et generaliter loco z^k scribatur $\frac{d^ky}{dx^k}$, ita simili
factoribus singulis aequationis algebraicae formabuntur aequationes
tiales, quae necessario in aequatione differentiali proposita continebu
ue ex quibus proinde valores particulares pro y reperientur. Sic si p
$q - pz$ fuerit divisor aequationis algebraicae, ex hoc per legem
ur haec aequatio differentialis

$$qy - \frac{pdy}{dx} = 0,$$

o integrata dat

$$y = \alpha e^{\frac{qx}{p}},$$

o est ea ipsa, quam ex eodem factore $pz - q$ elicuimus.

17. Hinc intelligitur, si habeatur divisor quicunque aequationis
braicae, puta $p + qz + rzz$, tum aequationem ex hoc divisore oriu

$$py + \frac{qdy}{dx} + \frac{rddy}{dx^2} = 0$$

e valorem pro y, qui etiam satisfacit aequationi differentiali propo
hoc ergo illam difficultatem tollere poterimus, quae locum habet, si aeq
braica habeat duos pluresve factores aequales. Sit igitur $(p - qz)^2$ d

aequationis algebraicae, ex hocque evoluto resultabit haec aeq
differentialis

$$ppy - \frac{2pqdy}{dx} + \frac{qqddy}{dx^2} = 0.$$

Ponamus

$$y = e^{\frac{px}{q}} u,$$

factaque substitutione habebimus $ddu = 0$, hincque $u = \alpha$ +
factore quadrato $(p - qz)^2$ oritur sequens valor

$$y = e^{\frac{px}{q}} (\alpha + \beta x),$$

qui duas constantes arbitrarias complectitur.

18. Si aequatio algebraica habeat divisorem cubicum
n aequatione differentiali proposita continebitur haec

$$p^3 y - \frac{3ppqdy}{dx} + \frac{3pqqddy}{dx^2} - \frac{q^3 d^3 y}{dx^3} = 0,$$

quae posito

$$y = e^{\frac{px}{q}} u$$

transmutabitur in hanc: $d^3 u = 0$; unde oritur $u = \alpha + \beta x$ +
aequationi propositae satisfaciet iste valor particularis

$$y = e^{\frac{px}{q}} (\alpha + \beta x + \gamma xx).$$

Simili modo si aequatio algebraica

$$0 = A + Bz + Cz^2 + Dz^3 + \ldots + Nz^n$$

habeat divisorem biquadratum $(p - qz)^4$, tum ex eo nascetur
particularis satisfaciens

$$y = e^{\frac{px}{q}} (\alpha + \beta x + \gamma xx + \delta x^3).$$

Atque generaliter si divisor sit $(p - qz)^k$, erit valor inde ortus

$$y = e^{\frac{px}{q}} (\alpha + \beta x + \gamma xx + \delta x^3 + \ldots + \varkappa x^{k-1}),$$

ita ut is k constantes imaginarias involvat.

$$0 = A + Bz + Cz^2 + Dz^3 + \ldots\ldots + Nz^n$$

educantur valores pro y, qui aequationi propositae

$$0 = Ay + \frac{Bdy}{dx} + \frac{Cddy}{dx^2} + \frac{Dd^3y}{dx^3} + \ldots\ldots + \frac{Nd^ny}{dx^n}$$

iunt], hoc dubium ex natura rei facile tolli poterit. Sit divisor u compositus

$$p + qz + rzz + sz^3 + \text{etc.}$$

o formetur aequatio

$$0 = py + \frac{qdy}{dx} + \frac{rddy}{dx^2} + \frac{sd^3y}{dx^3} + \text{etc.}$$

atebit valorem completum ipsius y pro hac aequatione prodire, alores ipsius y, quos divisores simplices aequationis

$$0 = p + qz + rzz + sz^3 + \text{etc.}$$

tant, in unam summam colligantur; at divisores simplices huius aequa mul sunt divisores simplices illius

$$0 = A + Bz + Cz^2 + Dz^3 + \ldots\ldots + Nz^n;$$

ob rem valor ipsius y ex illo factore composito ortus, simul est valo ens aequationis propositae differentialis

$$0 = Ay + \frac{Bdy}{dx} + \frac{Cddy}{dx^2} + \frac{Dd^3y}{dx^3} + \ldots\ldots + \frac{Nd^ny}{dx^n}.$$

Inventis autem valoribus ipsius y, qui ex aliquot divisoribus sim inter se aequalibus aequationis

$$0 = A + Bz + Cz^2 + Dz^3 + \ldots\ldots + Nz^n$$

, altera difficultas solvenda restat, si haec aequatio habeat radice rias. Constat autem, si quaepiam aequatio habeat radices imaginarias numerum semper esse parem; atque ego alibi ostendi has radices ima perpetuo binis coniungendis in eiusmodi paria dispesci posse, quarun

tam summa quam productum fiat quantitas realis. Hinc loc
ginariorum prodibunt divisores compositi duarum dimension

$$p - qz + rzz$$

reales, qui autem divisores simplices habeant imaginarios.
divisore composito $qq < 4\,pr$; unde

$$\frac{q}{2\sqrt{pr}} < 1.$$

Posito ergo sinu toto $= 1$ erit $\frac{q}{2\sqrt{pr}}$ cosinus cuiuspiam anguli re
fietque[1])

$$q = 2\sqrt{pr} \cdot \cos A \cdot \varphi,$$

ex quo generalis forma divisorum compositorum, qui divisores i
contineant, erit huiusmodi

$$p - 2z\sqrt{pr} \cdot \cos A \cdot \varphi + rzz.$$

21. Sit igitur aequationis

$$0 = A + Bz + Cz^2 + \text{etc.}$$

eiusmodi divisor

$$p - 2z\sqrt{pr} \cdot \cos A \cdot \varphi + rzz;$$

ex quo inveniri debeat conveniens valor ipsius y. At ex hoc div
ista aequatio differentio-differentialis

$$0 = py - \frac{2\,dy\sqrt{pr}}{dx} \cos A \cdot \varphi + \frac{rddy}{dx^2},$$

ad quam integrandam ponatur

$$y = e^{fx \cos A \cdot \varphi}\, u$$

posito brevitatis gratia $f = \sqrt{\frac{p}{r}}$ fietque

$$ffu\,dx^2\ (\sin A \cdot \varphi)^2 + ddu = 0.$$

Multiplicetur per $2\,du$ et integretur, erit

$$ffuu\,dx^2 (\sin A \cdot \varphi)^2 + du^2 = a^2 ff\,dx^2 (\sin A \cdot \varphi)^2,$$

1) Vide notam 1 p. 107 huius voluminis.

$$\int dx \sin A \cdot \varphi = \frac{du}{\sqrt{(a^2 - u^2)}};$$

ntegrata dat

$$\int x \sin A \cdot \varphi + \beta = A \sin \cdot \frac{u}{a} \cdot$$

a aequatione fit

$$u = a \sin A \cdot (\int x \sin A \cdot \varphi + \beta).$$

quenter habetur

$$y = a e^{\int x \cos A \cdot \varphi} \sin A \cdot (\int x \sin A \cdot \varphi + \beta),$$

t valor conveniens ipsius y pro aequatione proposita.

2. Eadem vel aequivalens expressio pro y colligitur ex factoribu
cibus etsi imaginariis aequationis

$$0 = p - 2z\sqrt{pr} \cdot \cos A \cdot \varphi + rzz,$$

posito $f = \sqrt{\frac{p}{r}}$ abit in hanc

$$0 = ff - 2fz \cos A \cdot \varphi + zz,$$

radices sunt

$$z = f \cos A \cdot \varphi \pm f\sqrt{-1} \cdot \sin A \cdot \varphi.$$

pro y resultant valores

$$e^{fx \cos A \cdot \varphi + fx\sqrt{-1} \cdot \sin A \cdot \varphi} \text{ et } e^{fx \cos A \cdot \varphi - fx\sqrt{-1} \cdot \sin A \cdot \varphi}$$

s coniunctis fit

$$y = e^{fx \cos A \cdot \varphi} (\eta e^{+fx\sqrt{-1} \cdot \sin A \cdot \varphi} + \theta e^{-fx\sqrt{-1} \cdot \sin A \cdot \varphi}).$$

utem exponentialibus in series conversis prodibit

$$y = e^{fx \cos A \cdot \varphi} \begin{cases} (\eta + \theta)\left(1 - \frac{ffxx(\sin A \cdot \varphi)^2}{1 \cdot 2} + \frac{f^4x^4(\sin A \cdot \varphi)^4}{1 \cdot 2 \cdot 3 \cdot 4} + \text{etc.}\right) \\ (\eta - \theta)\sqrt{-1} \cdot \left(fx \sin A \cdot \varphi - \frac{f^3x^3(\sin A \cdot \varphi)^3}{1 \cdot 2 \cdot 3} + \text{etc.}\right) \end{cases}$$

o ergo

$$\eta + \theta = a \text{ et } (\eta - \theta)\sqrt{-1} = \beta$$

quae expressio ad priorem facile reducitur.

23. Hine adipiscimur modum inveniendi v
pluresve huiusmodi divisores compositi fuerint in

$$(ff - 2fz \cos A \cdot \varphi + zz)^2$$

divisor aequationis algebraicae; quoniam is reducitu

$$(z - f \cos A \cdot \varphi - f \sqrt{-1} \cdot \sin A \cdot \varphi)^2 \quad (z - f \cos A$$

erit per praecedentia valor ipsius y hinc oriundus:

$$y = e^{fx \cos A \cdot \varphi + fx \sqrt{-1} \cdot \sin A \cdot \varphi} (\eta + \theta x) + e^{fx \cos A \cdot \varphi}$$

Cum autem sit

$$e^{+fx \sqrt{-1} \cdot \sin A \cdot \varphi} \eta + e^{-fx \sqrt{-1} \cdot \sin A}$$
$$= \alpha \cdot \cos A \cdot fx \sin A \cdot \varphi + \beta \sin A \cdot f$$

hinc colligitur fore

$$y = e^{fx \cos A \cdot \varphi} [(\alpha + \beta x) \cos A \cdot fx \sin A \cdot \varphi + (\gamma +$$

24. Quod si autem cubus aliave potestas ipsiu

$$ff - 2fz \cos A \cdot \varphi + zz$$

fuerit divisor aequationis algebraicae

$$0 = A + Bz + Czz + Dz^3 + \ldots$$

tum ex potestatibus iisdem factorum simplicium im
y eruantur secundum § 18 et in unam summam con
titates exponentiales imaginariae in sinus et cos
converti possunt ope huius lemmatis

$$e^{+fx \sqrt{-1} \cdot \sin A \cdot \varphi} \eta x^k + e^{-fx \sqrt{-1} \cdot \sin A}$$
$$= \alpha x^k \cos A \cdot fx \sin A \cdot \varphi + \beta x^k \sin A$$

Sic si

$$(ff - 2fz \cos A \cdot \varphi + zz)$$

$$y = e^{fx \cos A \cdot \varphi} [(\alpha + \beta x + \gamma x^2 + \delta x^3) \cos A \cdot fx \sin A \cdot \varphi$$
$$+ (\varepsilon + \zeta x + \eta x^2 + \theta x^3) \sin A \cdot fx \sin A \cdot \varphi].$$

25. Expressiones istae pluribus modis immutari possunt, prout antes aliis atque aliis modis exprimuntur. Commodissima autem vi ec transmutatio, qua valores ipsius y ad formam § 21 inventam reduci haec forma

$$\mu x^k \cos A \cdot fx \sin A \cdot \varphi + \nu x^k \sin A \cdot fx \sin A \cdot \varphi,$$

ponatur

$$\mu = \lambda \sin A \cdot p, \text{ et } \nu = \lambda \cos A \cdot p,$$

nsmutabitur in hanc

$$\lambda x^k \sin A \cdot (fx \sin A \cdot \varphi + p).$$

uamobrem ex factore indefiniti exponentis

$$(ff - 2fz \cos A \cdot \varphi + zz)^k$$

rmabitur sequens valor ipsius y:

$$y = e^{fx \cos A \cdot \varphi} (\alpha \sin A \cdot (fx \sin A \cdot \varphi + \mathfrak{A}) + \beta x \sin A \cdot (fx \sin A \cdot \varphi + \mathfrak{B}$$
$$+ \gamma x^2 \sin A \cdot (fx \sin A \cdot \varphi + \mathfrak{C}) + \ldots\ldots + \varkappa x^{k-1} \sin A \cdot (fx \sin A \cdot \varphi +$$

eque pacto ex omnibus divisoribus, utcunque fuerint comparati, va ales pro variabili y inveniuntur.

26. Quod iam ad constantes arbitrarias, quae in valores ipsius y odo inveniendos ingrediuntur, attinet, patet primo ex factoribus simpli rmae $f - z$ oriri valores ipsius y unicam constantem arbitrariam contine nde valor ipsius y, qui oritur ex factore $(f - z)^k$, continet k const bitrarias. Porro ex factore composito

$$ff - 2fz \cos A \cdot \varphi + zz$$

odit valor ipsius y duas constantes arbitrarias complectens; atque ex h odi factorum potestate quacunque

$$(ff - 2fz \cos A \cdot \varphi + zz)^k$$

formatur valor ipsius y, in quo $2k$ constantes arbitrariae ... constantium arbitrariarum aequalis sit numero dimens... haec variabilis in divisore obtinet, ex quo valor ipsius y ...

27. Quodsi ergo aequatio algebraica, quam ex a... proposita formavimus,

$$0 = A + Bz + Cz^2 + Dz^3 + Ez^4 + \dots$$

in factores suos sive simplices sive compositos reales siv... states simplicium compositorumve, resolvatur atque... singulis valores convenientes ipsius y formentur, tum hi ... iunctim considerati tot continebunt constantes arbitrari... n insunt unitates. Omnes igitur isti valores in unam ... solum valorem praebebunt pro y, qui aequationi propos...

$$0 = Ay + \frac{Bdy}{dx} + \frac{Cddy}{dx^2} + \frac{Dd^3y}{dx^3} + \dots$$

satisfaciat, verum etiam iste ipsius y erit valor completus... valores huic aequationi convenientes in se complecte... aequatio ista differentialis perfecte integratur in termin... grale unquam alias praeter hyperbolae atque circuli qua...

PROBLEMA I

28. Si proposita fuerit aequatio differentialis grad...

$$0 = Ay + \frac{Bdy}{dx} + \frac{Cddy}{dx^2} + \frac{Dd^3y}{dx^3} + \dots$$

in qua elementum dx positum est constans, ac littera... denotant coefficientes constantes quoscunque: inveni... integrale in terminis finitis realibus.

Solutio

Scribatur 1 loco y, z loco $\frac{dy}{dx}$, z^2 loco $\frac{ddy}{dx^2}$ et ge... hincque formetur sequens aequatio algebraica n dimensio...

$$0 = A + Bz + Cz^2 + Dz^3 + \dots +$$

ores compositi reales, in quibus z habeat duas dimensiones, q
nper bini factores imaginarii unum factorem compositum realem
unnt. Ex singulis divisoribus deinceps formentur sequenti modo v
rticulares pro y. Ex factore scilicet quolibet simplici, qui alios non ha
uales, huius formae $f - z$ oritur iste valor

$$y = \alpha e^{fx}.$$

duobus autem pluribusve factoribus aequalibus coniunctim sumtis v
ius y determinari debent. Nempe ex factore $(f - z)^2$ oritur

$$y = (\alpha + \beta x)\, e^{fx};$$

factore $(f - z)^3$ oritur

$$y = (\alpha + \beta x + \gamma xx)\, e^{fx};$$

que generaliter ex factore $(f - z)^k$ deducitur

$$y = e^{fx}\,(\alpha + \beta x + \gamma xx + \ldots\ldots + \varkappa x^{k-1}).$$

uod ad factores compositos attinet, si illa aequatio algebraica habe
rem

$$ff - 2fz \cos A \cdot \varphi + zz,$$

i sui similem inter reliquos non habeat, erit valor ex eo oriundus

$$y = e^{fx \cos A \cdot \varphi}\, \alpha \sin A \cdot (fx \sin A \cdot \varphi + \mathfrak{A}).$$

aequatio algebraica duos huiusmodi factores habeat aequales, ita
visibilis per

$$(ff - 2fz \cos A \cdot \varphi + zz)^2,$$

m ex divisore quadrato oritur sequens valor

$$= \alpha e^{fx \cos A \cdot \varphi} \sin A \cdot (fx \sin A \cdot \varphi + \mathfrak{A}) + \beta x e^{fx \cos A \cdot \varphi} \sin A \cdot (fx \sin A \cdot \varphi$$

n autem huius factoris potestas quaecunque puta

$$(ff - 2fz \cos A \cdot \varphi + zz)^k$$

erit divisor aequationis algebraicae, tum ex eo resultat sequens valor

$$= \alpha e^{fx \cos A \cdot \varphi} \sin A \cdot (fx \sin A \cdot \varphi + \mathfrak{A}) + \beta x e^{fx \cos A \cdot \varphi} \sin A \cdot (fx \sin A \cdot \varphi$$
$$\gamma x^2 e^{fx \cos A \cdot \varphi} \sin A \cdot (fx \sin A \cdot \varphi + \mathfrak{C}) + \delta x^3 e^{fx \cos A \cdot \varphi} \sin A \cdot (fx \sin A \cdot \varphi$$
$$+ \ldots\ldots + \varkappa x^{k-1} e^{fx \cos A \cdot \varphi} \sin A \cdot (fx \sin A \cdot \varphi + \mathfrak{K}).$$

atque is ipse, qui proditurus esset, si aequatio differentialis
n vicibus integraretur. Q. E. I.

Exemplum 1

29. Huius aequationis differentialis secund

$$0 = ay + \frac{bdy}{dx} + \frac{cddy}{dx^2}$$

integrale invenire.

Positis uti praecepimus 1 pro y, z pro $\frac{dy}{dx}$ et zz p
aequatio

$$0 = a + bz + czz;$$

quae vel ambas radices habebit reales, vel imaginarias; priu
posterius si $bb < 4ac$. Sit igitur primo $bb > 4ac$, ac duae

$$z = \frac{-b \pm \sqrt{(bb - 4ac)}}{2c}$$

hocque casu erit integrale quaesitum

$$y = \alpha e^{\frac{-bx + x\sqrt{(bb-4ac)}}{2c}} + \beta e^{\frac{-bx - x\sqrt{(bb-4ac)}}{2c}}.$$

Casus hic seorsim est perpendendus, quo $bb = 4ac$, tum e

$$a + 2z\sqrt{ac} + czz$$

quadratum nempe

$$(\sqrt{a} + z\sqrt{c})^2,$$

quod comparatum cum forma $(f - z)^2$ dat

$$f = -\sqrt{\frac{a}{c}},$$

ex quo integrale erit

$$y = (\alpha + \beta x)\, e^{-x\sqrt{\frac{a}{c}}},$$

$bb < 4ac$, atque aequatio

$$0 = a + bz + czz$$

ebit radices reales, comparata ergo cum forma

$$ff - 2fz \cos A \cdot \varphi + zz$$

$$\frac{b}{c} = -2f \cos A \cdot \varphi \text{ et } \frac{a}{c} = ff;$$

it

$$f = \sqrt{\frac{a}{c}} \text{ et } \cos A \cdot \varphi = \frac{-b}{2\sqrt{ac}}$$

$$\sin A \cdot \varphi = \frac{\sqrt{(4ac - bb)}}{2\sqrt{ac}},$$

oritur integrale

$$y = \alpha e^{\frac{-bx}{2c}} \sin A \cdot \left(\frac{x\sqrt{(4ac - bb)}}{2c} + \mathfrak{A}\right).$$

Exemplum 2

Huius aequationis differentialis tertii gradus

$$0 = y - \frac{3a^2 ddy}{dx^2} + \frac{2a^3 d^3y}{dx^3}$$

ale invenire.

hac aequatione ergo oritur ista algebraica

$$0 = 1 - 3a^2 zz + 2a^3 z^3,$$

solvitur in hos factores

$$(1 + 2az), \ (1 - az)^2.$$

ctor $1 + 2az$ cum forma $f - z$ comparatus dat

$$f = \frac{-1}{2a}$$

posterior factor $(1 - az)^2$ comparari debet cum $(f - z)^2$, ex

$$f = \frac{1}{a},$$

hincque nascitur

$$y = (\beta + \gamma x)\, e^{\frac{x}{a}}.$$

Aequationis ergo propositae integrale completum erit

$$y = \alpha e^{\frac{-x}{2a}} + (\beta + \gamma x)\, e^{\frac{x}{a}}.$$

Exemplum 3

31. Huius aequationis differentialis tertii gr

$$0 = y - \frac{a^3 d^3 y}{dx^3}$$

integrale invenire.

Aequatio algebraica ex hac aequatione orta erit

$$0 = 1 - a^3 z^3,$$

quae resolvitur in hos factores:

$$(1 - az),\ (1 + az + a^2 zz)$$

ita ut eius divisores sint hi

$$\frac{1}{a} - z \text{ et } \frac{1}{aa} + \frac{z}{a} + zz,$$

quorum iste in simplices reales resolvi nequit. Ille igitur div integrali

$$y = \alpha e^{\frac{x}{a}},$$

alter vero divisor

$$\frac{1}{aa} + \frac{z}{a} + zz$$

cum forma

$$ff - 2fz \cos A \cdot \varphi + zz$$

it fiat

$$\cos A \cdot \varphi = \frac{-1}{2} \text{ et } \sin A \cdot \varphi = \frac{\sqrt{3}}{2};$$

ex isto divisore resultat

$$y = \beta e^{\frac{-x}{2a}} \sin A \cdot \left(\frac{x\sqrt{3}}{2a} + \mathfrak{A}\right).$$

uationis ergo propositae integrale completum erit

$$y = \alpha e^{\frac{x}{a}} + \beta e^{\frac{-x}{2a}} \sin A \cdot \left(\frac{x\sqrt{3}}{2a} + \mathfrak{A}\right).$$

Exemplum 4

32. Huius aequationis differentialis quarti gradus

$$0 = y - \frac{a^4 d^4 y}{dx^4}$$

egrale invenire.

Ex hac aequatione formabitur ista aequatio algebraica

$$0 = 1 - a^4 z^4,$$

e duos habet divisores simplices reales

$$\frac{1}{a} - z \text{ et } \frac{1}{a} + z,$$

qui duo imaginarii continentur in hoc composito

$$\frac{1}{aa} + zz.$$

divisores simplices pro integrali dant

$$y = \alpha e^{\frac{x}{a}} + \beta e^{\frac{-x}{a}}.$$

isor autem

$$\frac{1}{aa} + zz$$

cum forma

$$ff - 2fz \cos A \cdot \varphi + zz$$

comparatus dat

$$f = \frac{1}{a} \text{ et } \cos A \cdot \varphi = 0,$$

hincque

$$\sin A \cdot \varphi = 1.$$

Terminus ergo exponentialis

$$e^{fx \cos A \cdot \varphi}$$

ob exponentem $= 0$ abit in unitatem, eritque

$$y = \gamma \sin A \cdot \left(\frac{x}{a} + \mathfrak{A}\right).$$

Integrale ergo completum erit:

$$y = \alpha e^{\frac{x}{a}} + \beta e^{\frac{-x}{a}} + \gamma \sin A \cdot \left(\frac{x}{a} + \mathfrak{A}\right).$$

Exemplum 5

33. Huius aequationis differentialis quarti g

$$0 = y + \frac{a^4 d^4 y}{dx^4}$$

integrale invenire.

Resolvi ergo oportebit istam aequationem algebraica

$$0 = 1 + a^4 z^4,$$

quae cum nullum habeat divisorem simplicem realem, resol factores compositos reales

$$1 + az\sqrt{2} + aazz \text{ et } 1 - az\sqrt{2} + aaz$$

qui divisi per aa, ut cum forma

$$ff - 2fz \cos A \cdot \varphi + zz$$

comparari queant, dabunt

$$\frac{1}{aa} + \frac{z\sqrt{2}}{a} + zz \text{ et } \frac{1}{aa} - \frac{z\sqrt{2}}{a} + zz;$$

hac

$$/ \cos A \cdot \varphi = \frac{1}{a\sqrt{2}};$$

iterum pro utraque

$$/ \sin A \cdot \varphi = \frac{1}{a\sqrt{2}}.$$

...us oritur integrale completum aequationis propositae

$$y = \alpha e^{\frac{-x}{a\sqrt{2}}} \sin A \cdot \left(\frac{x}{a\sqrt{2}} + \mathfrak{A}\right) + \beta e^{\frac{x}{a\sqrt{2}}} \sin A \cdot \left(\frac{x}{a\sqrt{2}} + \mathfrak{B}\right).$$

Exemplum 6

Huius aequationis differentialis septimi gradus

$$0 = y + \frac{ddy}{dx^2} + \frac{d^3y}{dx^3} + \frac{d^4y}{dx^4} + \frac{d^5y}{dx^5} + \frac{d^7y}{dx^7}$$

...ale completum invenire.

...scitur hinc ista aequatio algebraica septimi ordinis

$$0 = 1 + zz + z^3 + z^4 + z^6 + z^7,$$

...solvitur in sequentes factores reales tam simplices quam compositos

$$(1 + z),\ (1 + z + zz),\ (1 - z + zz)^2.$$

... primus cum forma $/ - z$ comparatus dat $/ = -1$, hincque oritu...

$$y = \alpha e^{-x}.$$

...autem alter $1 + z + zz$ comparatus cum

$$// - 2/z \cos A \cdot \varphi + zz$$

$$/ = 1 \text{ et } \cos A \cdot \varphi = -\frac{1}{2},$$

$$\sin A \cdot \varphi = \frac{\sqrt{3}}{2},$$

et integrale hinc natum

$$y = \beta e^{\frac{-x}{2}} \sin A \cdot \left(\frac{x\sqrt{3}}{2} + \mathfrak{A}\right).$$

Tertius factor $(1 - z + zz)^2$ comparari debet cum forma

$$(ff - 2fz \cos A \cdot \varphi + zz)^2,$$

unde fit

$$f = 1, \ \cos A \cdot \varphi = \frac{1}{2} \ \text{ et } \ \sin A \cdot \varphi = \frac{\sqrt{3}}{2}$$

Ex eo igitur prodit integrale

$$y = \gamma e^{\frac{x}{2}} \sin A \cdot \left(\frac{x\sqrt{3}}{2} + \mathfrak{B}\right) + \delta x e^{\frac{x}{2}} \sin A \cdot \left(\frac{x\sqrt{3}}{2}\right.$$

Quamobrem aequationis differentialis propositae integra

$$y = \alpha e^{-x} + \beta e^{\frac{-x}{2}} \sin A \cdot \left(\frac{x\sqrt{3}}{2} + \mathfrak{A}\right)$$
$$+ \gamma e^{\frac{x}{2}} \sin A \cdot \left(\frac{x\sqrt{3}}{2} + \mathfrak{B}\right) + \delta x e^{\frac{x}{2}} \sin A \cdot \left(\frac{x\sqrt{3}}{2}\right.$$

in quo utique septem constantes arbitrariae continentur.

Exemplum 7

35. Huius aequationis differentialis octavi gra

$$0 = \frac{d^3y}{dx^3} - \frac{3\,d^4y}{dx^4} + \frac{4\,d^5y}{dx^5} - \frac{4\,d^6y}{dx^6} + \frac{3\,d^7y}{dx^7} - \frac{d^8}{dx}$$

integrale completum invenire.

Aequatio algebraica octavi gradus, quam resolvi oportet

$$0 = z^3 - 3z^4 + 4z^5 - 4z^6 + 3z^7 - z^8;$$

quam primum divisibilem esse constat per z^3, qui divisor cu comparatus dat $f = 0$, hincque pro integrali invenitur

$$y = \alpha + \beta x + \gamma xx.$$

Divisore hoc in computum ducto superest resolvenda haec aeq

$$0 = 1 - 3z + 4zz - 4z^3 + 3z^4 - z^5,$$

rato fit

$$f = 1 \text{ et } \cos A \cdot \varphi = 0,$$

sin $A \cdot \varphi = 1$; ideoque resultat

$$y = \delta \sin A \cdot (x + \mathfrak{A}).$$

one porro per $1 + zz$ instituta remanet aequatio

$$1 - 3z + 3zz - z^3 = 0 = (1 - z)^3;$$

na ergo $(f - z)^3$ fit $f = 1$, atque integrale hinc oriundum est

$$y = (\varepsilon + \zeta x + \eta xx)\, e^x.$$

quenter completum integrale aequationis propositae est

$$y = \alpha + \beta x + \gamma xx + \delta \sin A \cdot (x + \mathfrak{A}) + (\varepsilon + \zeta x + \eta xx)\, e^x.$$

Exemplum 8

6. Huius aequationis differentialis indefiniti gradus

$$0 = \frac{d^n y}{dx^n}$$

rale invenire.

esultat ista aequatio algebraica

$$z^n = 0,$$

um omnes radices sint aequales, ea comparari debet cum factore $(f - z$
$k = n$ et $f = 0$, ex quo statim prodit integrale quaesitum

$$y = \alpha + \beta x + \gamma x^2 + \delta x^3 + \ldots\ldots + \nu x^{n-1}.$$

ero idem integrale facile invenitur integrationem n vicibus success
endo. Prima enim integratione oritur

$$\alpha = \frac{d^{n-1} y}{dx^{n-1}};$$

licetur per dx et integretur secundo, erit

$$\alpha x + \beta = \frac{d^{n-2} y}{dx^{n-2}}.$$

$$\frac{}{2} + px + \gamma = \frac{}{dx^{n-3}}.$$

Atque ita porro, si integratio n vicibus repetatur, prodibit n
tium expressionibus id ipsum integrale, quod per nostram regu

37. Huius methodi beneficio possunt etiam plurimae al differentiales gradus indefiniti integrari, quae quidem ad ae braicas deducunt, quarum factores reales sive simplices sivo exhiberi possunt. Cum autem huius loci non sit modum tra huiusmodi aequationum indefiniti dimensionum numeri in eiusmodi aequationes differentiales insuper tractabimus, quae algebraicas perducunt, quarum factores iam aliunde sunt cogr aequationes autem sunt

$$f^n \pm z^n = 0 \text{ et } f^{2n} \pm 2pf^n z^n \pm z^{2n} = 0;$$

harum enim expressionum factores reales tam simplices quam trinomiales omnes exhibiti sunt a Viris de Analysi meritissi Moivraeo[1]), quos proinde tanquam cognitos in solutione seque matum assumemus.

PROBLEMA II

38. Si proposita fuerit ista aequatio differentialis gradus n

$$0 = y - \frac{d^n y}{dx^n},$$

in qua elementum dx ponitur constans, eius integrale completum

Solutio

Posito uti praescripsimus 1 loco y et z^n loco $\frac{d^n y}{dx^n}$ habebitur algebraica

$$0 = 1 - z^n,$$

1) Roger Cotes (1682—1716). Abraham de Moivre (1667—1754).

gināria, atque continentur in hac forma generali

$$1 - 2z\cos A \cdot \frac{2k\pi}{n} + zz$$

(π denotat semicircumferentiam circuli, cuius radius $= 1$); qui cum di[visore tri]nomiali generali

$$ff - 2fz\cos A \cdot \varphi + zz$$

[com]paratus dat

$$f = 1 \text{ et } \varphi = \frac{2k\pi}{n},$$

[eri]t hic divisor det valorem integralem

$$y = ae^{x\cos A \cdot \frac{2k\pi}{n}} \sin A \cdot \left(x \sin A \cdot \frac{2k\pi}{n} + \mathfrak{A}\right).$$

[Qu]odsi iam loco $2k$ successive omnes numeri pares exponentem n non [exced]entes substituantur, prodibunt omnes possibiles valores, qui pro y [aeq]uati satisfaciunt. Continetur vero etiam in hac generali forma valor i[lle], qui oritur ex factore simplici $1 - z$, qui est $y = ae^x$; posito enim $k =$ [0 fit]

$$\cos A \cdot \frac{2k\pi}{n} = 1 \text{ et } \sin A \cdot \frac{2k\pi}{n} = 0$$

[ide]oque $y = ae^x$, ob $\sin A \cdot \mathfrak{A}$ constantem in a complexum. Simili modo [si n fuerit] numerus par, valor ipsius y ex factore $1 + z$ oriundus, qui est $y = ae^{-x}$, [ex fact]ore generali resultat facto $2k = n$, fit enim tum

$$\cos A \cdot \frac{2k\pi}{n} = -1 \text{ et } \sin A \cdot \frac{2k\pi}{n} = 0,$$

[und]e fit valor ex factore generali oriundus $y = ae^{-x}$. Integrale ergo compl[etum obti]nebitur, si in forma generali

$$y = ae^{x\cos A \cdot \frac{2k\pi}{n}} \sin A \cdot \left(x \sin A \cdot \frac{2k\pi}{n} + \mathfrak{A}\right)$$

[succ]essive loco $2k$ omnes numeri pares a 0 usque ad n substituantur [et om]nes valores in unam summam coniiciantur. Prodibit ergo integrale quaesitu[m com]pletum

$$+ \gamma e \quad \sin A \cdot \Big(x \sin A \cdot \frac{\quad}{n} +$$

$$+ \delta e^{x \cos A \cdot \frac{6\pi}{n}} \sin A \cdot \Big(x \sin A \cdot \frac{6\pi}{n} +$$

$$+ \varepsilon e^{x \cos A \cdot \frac{8\pi}{n}} \sin A \cdot \Big(x \sin A \cdot \frac{8\pi}{n} +$$

quae membra eousque debent continuari, quoad n c
habeantur, vel quod eodem redit, quoad coefficiens ip
evadat. Fiet autem, si n sit numerus impar, ultimum me

$$= \nu e^{x \cos A \cdot \frac{(n-1)\pi}{n}} \sin A \cdot \Big(x \sin A \cdot \frac{(n-1)\pi}{n}$$

at si n sit numerus par, erit ultimum membrum $= \nu\, e^{-x}$

$$= \mu e^{x \cos A \cdot \frac{(n-2)\pi}{n}} \sin A \cdot \Big(x \sin A \cdot \frac{(n-2)\pi}{n}$$

Pro quovis ergo valore ipsius n integrale completum
Q. E. I.

39. Quo ista integralia clarius ob oculos ponantur,
ipsius n ab unitate incipiendo integralia aequationis

$$0 = y - \frac{d^n y}{dx^n}$$

exhibeamus:

I. Huius aequationis $0 = y - \frac{dy}{dx}$ integrale est:

$$y = \alpha e^x$$

II. Huius aequationis $0 = y - \frac{ddy}{dx^2}$ integrale est:

$$y = \alpha e^x + \beta e^{-x}$$

$$y = \alpha e^x + \beta e^{x \cos A \cdot \frac{2}{3}\pi} \sin A \cdot \left(x \sin A \cdot \frac{2}{3}\pi + \mathfrak{B}\right)$$

Huius aequationis $0 = y - \frac{d^4 y}{dx^4}$ integrale est:

$$y = \alpha e^x + \beta \sin A \cdot (x + \mathfrak{B}) + \gamma e^{-x}$$

Huius aequationis $0 = y - \frac{d^5 y}{dx^5}$ integrale est:

$$y = \alpha e^x + \beta e^{x \cos A \cdot \frac{2}{5}\pi} \sin A \cdot \left(x \sin A \cdot \frac{2}{5}\pi + \mathfrak{B}\right)$$
$$+ \gamma e^{x \cos A \cdot \frac{4}{5}\pi} \sin A \cdot \left(x \sin A \cdot \frac{4}{5}\pi + \mathfrak{C}\right)$$

Huius aequationis $0 = y - \frac{d^6 y}{dx^6}$ integrale est:

$$y = \alpha e^x + \beta e^{x \cos A \cdot \frac{1}{3}\pi} \sin A \cdot \left(x \sin A \cdot \frac{1}{3}\pi + \mathfrak{B}\right)$$
$$+ \gamma e^{x \cos A \cdot \frac{2}{3}\pi} \sin A \cdot \left(x \sin A \cdot \frac{2}{3}\pi + \mathfrak{C}\right) + \delta e^{-x}$$

Huius aequationis $0 = y - \frac{d^7 y}{dx^7}$ integrale est:

$$y = \alpha e^x + \beta e^{x \cos A \cdot \frac{2}{7}\pi} \sin A \cdot \left(x \sin A \cdot \frac{2}{7}\pi + \mathfrak{B}\right)$$
$$+ \gamma e^{x \cos A \cdot \frac{4}{7}\pi} \sin A \cdot \left(x \sin A \cdot \frac{4}{7}\pi + \mathfrak{C}\right)$$
$$+ \delta e^{x \cos A \cdot \frac{6}{7}\pi} \sin A \cdot \left(x \sin A \cdot \frac{6}{7}\pi + \mathfrak{D}\right)$$

Huius aequationis $0 = y - \frac{d^8 y}{dx^8}$ integrale est:

$$= \alpha e^x + \beta e^{x \cos A \cdot \frac{1}{4}\pi} \sin A \cdot \left(x \sin A \cdot \frac{1}{4}\pi + \mathfrak{B}\right) + \gamma \sin A \cdot (x + \mathfrak{C})$$
$$+ \delta e^{x \cos A \cdot \frac{3}{4}\pi} \sin A \cdot \left(x \sin A \cdot \frac{3}{4}\pi + \mathfrak{D}\right) + \varepsilon e^{-x} \text{ etc.}$$

PROBLEMA

40. Si proposita fuerit ista aequatio differentialis gradus in

$$0 = y + \frac{d^n y}{dx^n},$$

posito elemento dx constante, eius integrale invenire.

Solutio

Posito secundum regulam 1 pro y et z^n pro $\frac{d^n y}{dx^n}$, prodibit i
algebraica $0 = 1 + z^n$, quae si n fuerit numerus impar, divisorem
realem habet $1 + z$, ex quo oritur $y = a e^{-x}$. Reliqui divisores sim
sunt imaginarii; horum vero bini continentur in hoc factore trin

$$1 - 2 z \cos A \cdot \frac{2k-1}{n} \pi + zz,$$

haecque expressio omnes prorsus divisores formae $1 + z^n$ sugg
$2k - 1$ omnes numeri impares ipso n non maiores successive su
Collata autem hac formula

$$1 - 2 z \cos A \cdot \frac{2k-1}{n} \pi + zz$$

cum factore generali

$$ff - 2 fz \cos A \cdot \varphi + zz$$

fit

$$f = 1 \text{ et } \varphi = \frac{2k-1}{n} \pi;$$

hinc ergo enascitur sequens pro y valor generalis

$$y = a e^{x \cos A \cdot \frac{2k-1}{n} \pi} \sin A \cdot \left(x \sin A \cdot \frac{2k-1}{n} \pi + \mathfrak{A}\right).$$

Atque in hoc valore generali etiam continetur valor ipsius y ex
plici $1 + z$, si quidem n fuerit numerus impar, oriundus; prodit en
$y = a e^{-x}$, si fiat $2k - 1 = n$, tum enim fit

$$\cos A \cdot \frac{2k-1}{n} \pi = \cos A \cdot \pi = -1$$

$$y = \alpha e^{x \cos A \cdot \frac{2k-1}{n} \pi} \sin A \cdot \left(x \sin A \cdot \frac{2k-1}{n} \pi + \mathfrak{A}\right)$$

k — 1 successive omnes numeri impares 1, 3, 5, 7 etc., qui quidem ex- e n non sunt maiores, substituantur istique valores cuncti in unam m colligantur. Prodibit ergo hoc modo integrale quaesitum et completum

$$\begin{aligned} y = {} & \alpha e^{x \cos A \cdot \frac{1}{n} \pi} \sin A \cdot \left(x \sin A \cdot \frac{1}{n} \pi + \mathfrak{A}\right) \\ & + \beta e^{x \cos A \cdot \frac{3}{n} \pi} \sin A \cdot \left(x \sin A \cdot \frac{3}{n} \pi + \mathfrak{B}\right) \\ & + \gamma e^{x \cos A \cdot \frac{5}{n} \pi} \sin A \cdot \left(x \sin A \cdot \frac{5}{n} \pi + \mathfrak{C}\right) \\ & + \delta e^{x \cos A \cdot \frac{7}{n} \pi} \sin A \cdot \left(x \sin A \cdot \frac{7}{n} \pi + \mathfrak{D}\right) + \text{etc.}, \end{aligned}$$

nembra eousque continuari debent, quoad n constantes arbitrariae ingressae; quod eveniet, si ex serie fractionum

$$\frac{1}{n}, \ \frac{3}{n}, \ \frac{5}{n}, \ \frac{7}{n} \text{ etc.}$$

unitatem non superantes capiantur. Fiet autem, si n sit numerus par, um ultimum

$$\nu e^{x \cos A \cdot \frac{n-1}{n} \pi} \sin A \cdot \left(x \sin A \cdot \frac{n-1}{n} \pi + \mathfrak{N}\right).$$

numerus impar, membrum ultimum erit:

$$\nu e^{-x},$$

mum vero

$$\mu e^{x \cos A \cdot \frac{n-2}{n} \pi} \sin A \cdot \left(x \sin A \cdot \frac{n-2}{n} \pi + \mathfrak{M}\right),$$

ullo negotio integrale completum quovis casu assignatur. Q. E. I.

... integrationis penitius perspici...

I. Huius aequationis $0 = y + \frac{dy}{dx}$ integrale est:

$$y = \alpha e^{-x}$$

II. Huius aequationis $0 = y + \frac{ddy}{dx^2}$ integrale est:

$$y = \alpha \sin A \cdot (x + \mathfrak{A})$$

III. Huius aequationis $0 = y + \frac{d^3y}{dx^3}$ integrale est:

$$y = \alpha e^{x \cos A \cdot \frac{1}{3}\pi} \sin A \cdot \left(x \sin A \cdot \frac{1}{3}\pi + \mathfrak{A}\right) + \beta e^{-x}$$

IV. Huius aequationis $0 = y + \frac{d^4y}{dx^4}$ integrale est:

$$y = \alpha e^{x \cos A \cdot \frac{1}{4}\pi} \sin A \cdot \left(x \sin A \cdot \frac{1}{4}\pi + \mathfrak{A}\right)$$
$$+ \beta e^{x \cos A \cdot \frac{3}{4}\pi} \sin A \cdot \left(x \sin A \cdot \frac{3}{4}\pi + \mathfrak{B}\right)$$

V. Huius aequationis $0 = y + \frac{d^5y}{dx^5}$ integrale est:

$$y = \alpha e^{x \cos A \cdot \frac{1}{5}\pi} \sin A \cdot \left(x \sin A \cdot \frac{1}{5}\pi + \mathfrak{A}\right)$$
$$+ \beta e^{x \cos A \cdot \frac{3}{5}\pi} \sin A \cdot \left(x \sin A \cdot \frac{3}{5}\pi + \mathfrak{B}\right) + \gamma e^{-x}$$

VI. Huius aequationis $0 = y + \frac{d^6y}{dx^6}$ integrale est:

$$y = \alpha e^{x \cos A \cdot \frac{1}{6}\pi} \sin A \cdot \left(x \sin A \cdot \frac{1}{6}\pi + \mathfrak{A}\right) + \beta \sin A \cdot (x +$$
$$+ \gamma e^{x \cos A \cdot \frac{5}{6}\pi} \sin A \cdot \left(x \sin A \cdot \frac{5}{6}\pi + \mathfrak{C}\right)$$

$$y = \alpha e^{x \cos A \cdot \frac{1}{7}\pi} \sin A \cdot \left(x \sin A \cdot \frac{1}{7}\pi + \mathfrak{A}\right)$$

$$+ \beta e^{x \cos A \cdot \frac{3}{7}\pi} \sin A \cdot \left(x \sin A \cdot \frac{3}{7}\pi + \mathfrak{B}\right)$$

$$+ \gamma e^{x \cos A \cdot \frac{5}{7}\pi} \sin A \cdot \left(x \sin A \cdot \frac{5}{7}\pi + \mathfrak{C}\right) + \delta e^{-x}$$

Huius aequationis $0 = y + \frac{d^8 y}{dx^8}$ integrale est:

$$y = \alpha e^{x \cos A \cdot \frac{1}{8}\pi} \sin A \cdot \left(x \sin A \cdot \frac{1}{8}\pi + \mathfrak{A}\right)$$

$$+ \beta e^{x \cos A \cdot \frac{3}{8}\pi} \sin A \cdot \left(x \sin A \cdot \frac{3}{8}\pi + \mathfrak{B}\right)$$

$$+ \gamma e^{x \cos A \cdot \frac{5}{8}\pi} \sin A \cdot \left(x \sin A \cdot \frac{5}{8}\pi + \mathfrak{C}\right)$$

$$+ \delta e^{x \cos A \cdot \frac{7}{8}\pi} \sin A \cdot \left(x \sin A \cdot \frac{7}{8}\pi + \mathfrak{D}\right)$$

PROBLEMA IV

. Si proposita fuerit aequatio differentialis gradus $2n$ haec:

$$0 = y + \frac{2h\,d^n y}{dx^n} + \frac{d^{2n} y}{dx^{2n}}$$

elemento dx constante, eius integrale invenire, existente $hh > 1$.

Solutio

secundum regulam ponamus 1 pro y, z^n pro $\frac{d^n y}{dx^n}$ et z^{2n} pro $\frac{d^{2n}}{dx^{2n}}$ ista aequatio algebraica

$$0 = 1 + 2hz^n + z^{2n},$$

b $hh > 1$ in hos duos factores resolvitur:

$$[\, z^n + h + \sqrt{(hh - 1)}\,]\,[\, z^n + h - \sqrt{(hh - 1)}\,].$$

erunt quantitates affirmativae. Sit ergo

$$h + \sqrt{(hh - 1)} = a^n \text{ et } h - \sqrt{(hh - 1)} = b^n,$$

ita ut sit $ab = 1$. Habebimus igitur istam aequationem in duos
resolutam:

$$0 = (z^n + a^n)\ (z^n + b^n)$$

atque prioris factoris singuli factores trinomiales reales contin
forma:

$$aa - 2az \cos A \cdot \frac{2k-1}{n}\pi + zz$$

posterioris vero in hac:

$$bb - 2bz \cos A \cdot \frac{2k-1}{n}\pi + zz.$$

Omnesque factores habebuntur, si in utraque forma successiv
ponantur omnes numeri impares 1, 3, 5, 7 etc., qui exponen
maiores. Integrale ergo quaesitum ex his factoribus ita formabit

$$y = Ae^{ax \cos A \cdot \frac{1}{n}\pi} \sin A \cdot \left(ax \sin A \cdot \frac{1}{n}\pi + \mathfrak{A}\right)$$

$$+ Be^{ax \cos A \cdot \frac{3}{n}\pi} \sin A \cdot \left(ax \sin A \cdot \frac{3}{n}\pi + \mathfrak{B}\right)$$

$$+ Ce^{ax \cos A \cdot \frac{5}{n}\pi} \sin A \cdot \left(ax \sin A \cdot \frac{5}{n}\pi + \mathfrak{C}\right)$$

$$+ De^{ax \cos A \cdot \frac{7}{n}\pi} \sin A \cdot \left(ax \sin A \cdot \frac{7}{n}\pi + \mathfrak{D}\right) +$$

$$+ \alpha e^{bx \cos A \cdot \frac{1}{n}\pi} \sin A \cdot \left(bx \sin A \cdot \frac{1}{n}\pi + \mathfrak{a}\right)$$

$$+ \beta e^{bx \cos A \cdot \frac{3}{n}\pi} \sin A \cdot \left(bx \sin A \cdot \frac{3}{n}\pi + \mathfrak{b}\right)$$

$$+ \gamma e^{bx \cos A \cdot \frac{5}{n}\pi} \sin A \cdot \left(bx \sin A \cdot \frac{5}{n}\pi + \mathfrak{c}\right) +$$

$$0 = y - \frac{2h\,d^n y}{dx^n} + \frac{d^{2n} y}{dx^{2n}}$$

constante et existente $hh > 1$, eius integrale invenire.

Solutio

[Sec]undum regulam supra datam orietur hic sequens aequatio algebraica

$$0 = 1 - 2hz^n + z^{2n};$$

[quae in] hos duos factores reales primum resolvitur:

$$0 = [z^n - h + \sqrt{(hh-1)}]\,[z^n - h - \sqrt{(hh-1)}].$$

[Cum aut]em h denotet quantitatem positivam, ponatur

$$h + \sqrt{(hh-1)} = a^n \text{ et } h - \sqrt{(hh-1)} = b^n,$$

[eritque] $ab = 1$; hincque orietur ista aequatio:

$$0 = (z^n - a^n)\,(z^n - b^n).$$

[Iam f]actoris $z^n - a^n$ omnes factores trinomiales reales continentur in [hac for]ma

$$aa - 2az \cos A \cdot \frac{2k}{n}\pi + zz;$$

[factor]is vero $z^n - b^n$ in hac forma

$$bb - 2bz \cos A \cdot \frac{2k}{n}\pi + zz;$$

[omnes]que factores habebuntur, si in utraque forma loco $2k$ successive ponantur [omnes n]umeri pares 0, 2, 4, 6 etc. numero n non maiores. Ex his itaque fac[toribus c]ognitis integrale quaesitum colligitur fore:

$$y = \left\{ \begin{array}{l} \quad + Ce^{\cdots} \sin A \cdot \left(ax \sin A \cdot \frac{\cdot}{n} \pi \right. \\ \quad + De^{ax \cos A \cdot \frac{6}{n}\pi} \sin A \cdot \left(ax \sin A \cdot \frac{6}{n} \pi \right. \\ + \alpha e^{bx} + \beta e^{bx \cos A \cdot \frac{2}{n}\pi} \sin A \cdot \left(bx \sin A \cdot \frac{2}{n} \pi \right. \\ \quad + \gamma e^{bx \cos A \cdot \frac{4}{n}\pi} \sin A \cdot \left(bx \sin A \cdot \frac{4}{n} \pi \right. \\ \quad + \delta e^{bx \cos A \cdot \frac{6}{n}\pi} \sin A \cdot \left(bx \sin A \cdot \frac{6}{n} \pi \right. \end{array} \right.$$

Q. E. I.

PROBLEMA VI

44. Si proposita fuerit aequatio differentialis grad

$$0 = y + \frac{2h\,d^n y}{dx^n} - \frac{d^{2n} y}{dx^{2n}}$$

sumto elemento dx constante, eius integrale invenire.

Solutio

Aequatio algebraica, quae secundum praecepta hinc

$$0 = 1 + 2hz^n - z^{2n}$$

in hos duos factores reales primum resolvitur:

$$0 = [h + \sqrt{(hh + 1)} - z^n]\ [-h + \sqrt{(hh}$$

Fiat, id quod ob h quantitatem positivam semper fieri po

$$\sqrt{(hh + 1)} + h = a^n \text{ et } \sqrt{(hh + 1)} -$$

ita ut fit $ab = 1$; hincque nascetur ista aequatio:

$$0 = (a^n - z^n)\ (b^n + z^n).$$

$$aa - 2az \cos A \cdot \frac{2k}{n}\pi + zz,$$

oris vero in hac:

$$bb - 2bz \cos A \cdot \frac{2k-1}{n}\pi + zz,$$

ue factores habebuntur, si in priori loco $2k$ omnes numeri par 6 etc., in posteriori vero loco $2k-1$ omnes impares 1, 3, 5, 7 et m n non excedentes successive substituantur. Ex his ergo factoribu integrale quaesitum colligitur:

$$y = \left\{\begin{aligned} & Ae^{ax} + Be^{ax\cos A\cdot\frac{2}{n}\pi}\sin A\cdot\left(ax\sin A\cdot\frac{2}{n}\pi+\mathfrak{B}\right) \\ & + Ce^{ax\cos A\cdot\frac{4}{n}\pi}\sin A\cdot\left(ax\sin A\cdot\frac{4}{n}\pi+\mathfrak{C}\right) \\ & + De^{ax\cos A\cdot\frac{6}{n}\pi}\sin A\cdot\left(ax\sin A\cdot\frac{6}{n}\pi+\mathfrak{D}\right) + \text{etc.} \\ & + \alpha e^{bx\cos A\cdot\frac{1}{n}\pi}\sin A\cdot\left(bx\sin A\cdot\frac{1}{n}\pi+\mathfrak{a}\right) \\ & + \beta e^{bx\cos A\cdot\frac{3}{n}\pi}\sin A\cdot\left(bx\sin A\cdot\frac{3}{n}\pi+\mathfrak{b}\right) \\ & + \gamma e^{bx\cos A\cdot\frac{5}{n}\pi}\sin A\cdot\left(bx\sin A\cdot\frac{5}{n}\pi+\mathfrak{c}\right) + \text{etc.} \end{aligned}\right.$$

Q. E. I.

PROBLEMA VII

. Si proposita fuerit aequatio differentialis gradus indefiniti $2n$ haec:

$$0 = y - \frac{2k\,d^ny}{dx^n} - \frac{d^{2n}y}{dx^{2n}},$$

positum est elementum dx constans, eius integrale invenire.

Solutio

r substitutionem per regulam supra datam faciendam nascitur hinc is io algebraica ordinis $2n$:

$$0 = [-h + \sqrt{(hh+1)} - z^n][h + \sqrt{(hh}$$

Ob h quantitatem positivam ponatur

$$\sqrt{(hh+1)} + h = a^n \text{ et } \sqrt{(hh+1)} - h$$

ita ut sit $ab = 1$. Atque sequens habebitur aequatio reso

$$0 = (a^n + z^n)(b^n - z^n),$$

cuius prioris factoris $a^n + z^n$ omnes factores trinomiales forma:

$$aa - 2az \cos A \cdot \frac{2k-1}{n} \pi + zz$$

posterioris vero in hac:

$$bb - 2bz \cos A \cdot \frac{2k}{n} \pi + zz;$$

omnesque factores habebuntur, si in illa forma ponant numeri impares 1, 3, 5, 7 etc. loco $2k-1$, in hac vero lo pares 0, 2, 4, 6 etc. numero n non maiores. Ex his itaque integrale quaesitum et completum:

$$y = \left\{ \begin{aligned} & \quad\quad\quad\;\; Ae^{ax \cos A \cdot \frac{1}{n}\pi} \sin A \cdot \left(ax \sin A \cdot \frac{1}{n}\pi \right. \\ & \quad\quad + Be^{ax \cos A \cdot \frac{3}{n}\pi} \sin A \cdot \left(ax \sin A \cdot \frac{3}{n}\pi \right. \\ & \quad\quad + Ce^{ax \cos A \cdot \frac{5}{n}\pi} \sin A \cdot \left(ax \sin A \cdot \frac{5}{n}\pi \right. \\ & + \alpha e^{bx} + \beta e^{bx \cos A \cdot \frac{2}{n}\pi} \sin A \cdot \left(bx \sin A \cdot \frac{2}{n}\pi \right. \\ & \quad\quad + \gamma e^{bx \cos A \cdot \frac{4}{n}\pi} \sin A \cdot \left(bx \sin A \cdot \frac{4}{n}\pi \right. \\ & \quad\quad + \delta e^{bx \cos A \cdot \frac{6}{n}\pi} \sin A \cdot \left(bx \sin A \cdot \frac{6}{n}\pi \right. \end{aligned} \right.$$

Q. E. I.

$$0 = y + \frac{2h\,d^n y}{dx^n} + \frac{d^{2n} y}{dx^{2n}}$$

sito elemento dx constante, et existente $hh < 1$, eius integrale comp
venire.

Solutio

Aequatio algebraica ordinis $2n$, quae hinc oritur, est

$$0 = 1 + 2hz^n + z^{2n},$$

cuius factores trinomiales reales omnes inveniendos capiatur in ci
us radius $= 1$, arcus ω, cuius cosinus sit $= h$, ita ut sit $h = \cos A \cdot \omega$.
c invento unusquisque factor trinomialis continebitur in hac forma:

$$1 - 2z \cos A \cdot \frac{k\pi - \omega}{n} + zz$$

ostituendo loco k omnes numeros impares $1, 3, 5, 7, \ldots, (2n-1)$,
rum factorum numerus futurus sit n, uti numerus dimensionum re
his igitur factoribus cognitis reperietur secundum praecepta data int
aesitum aequationis propositae:

$$\begin{aligned} y = {} & \alpha e^{x\cos A \cdot \frac{\pi-\omega}{n}} \sin A \cdot \left(x \sin A \cdot \frac{\pi-\omega}{n} + \mathfrak{a}\right) \\ & + \beta e^{x\cos A \cdot \frac{3\pi-\omega}{n}} \sin A \cdot \left(x \sin A \cdot \frac{3\pi-\omega}{n} + \mathfrak{b}\right) \\ & + \gamma e^{x\cos A \cdot \frac{5\pi-\omega}{n}} \sin A \cdot \left(x \sin A \cdot \frac{5\pi-\omega}{n} + \mathfrak{c}\right) \\ & + \text{etc.} \\ & + \nu e^{x\cos A \cdot \frac{(2n-1)\pi-\omega}{n}} \sin A \cdot \left(x \sin A \cdot \frac{(2n-1)\pi-\omega}{n} + \mathfrak{n}\right). \end{aligned}$$

merus scilicet membrorum hoc integrale constituentium est n, in
merus constantium arbitrariarum ingredientium est $2n$, uti gradus
ntialium aequationis propositae requirit. Q. E. I.

PROBLEMA IX

47. Existente iterum $hh < 1$, si proposita fuerit haec aequ
rentialis gradus indefiniti $2n$:

$$0 = y - \frac{2hd^n y}{dx^n} + \frac{d^{2n}y}{dx^{2n}}$$

sumto elemento dx constante, eius integrale completum invenire.

Solutio

Aequatio algebraica, quae secundum praecepta tradita hinc ded

$$0 = 1 - 2hz^n + z^{2n},$$

cuius singuli factores trinomiales reales, quorum numerus est n, c
in hac forma generali:

$$1 - 2z \cos A \cdot \frac{k\pi - \omega}{n} + zz,$$

si loco k successive omnes numeri pares 2, 4, 6, 8 etc. usque ad $2n$
substituantur. Denotat hic autem uti ante ω arcum circuli, cuiu
est h, qui ob $h < 1$ semper assignari potest, ita ut sit $h = \cos A \cdot \omega$
autem factoribus omnibus aequationis

$$0 = 1 - 2hz^n + z^{2n},$$

aequationis differentialis propositae integrale completum erit:

$$\begin{aligned}
y = {} & \alpha e^{x\cos A \cdot \frac{2\pi - \omega}{n}} \sin A \cdot \left(x \sin A \cdot \frac{2\pi - \omega}{n} + \mathfrak{a}\right) \\
& + \beta e^{x\cos A \cdot \frac{4\pi - \omega}{n}} \sin A \cdot \left(x \sin A \cdot \frac{4\pi - \omega}{n} + \mathfrak{b}\right) \\
& + \gamma e^{x\cos A \cdot \frac{6\pi - \omega}{n}} \sin A \cdot \left(x \sin A \cdot \frac{6\pi - \omega}{n} + \mathfrak{c}\right) \\
& + \delta e^{x\cos A \cdot \frac{8\pi - \omega}{n}} \sin A \cdot \left(x \sin A \cdot \frac{8\pi - \omega}{n} + \mathfrak{d}\right) \\
& + \text{etc.} \\
& + \nu e^{x\cos A \cdot \frac{2n\pi - \omega}{n}} \sin A \cdot \left(x \sin A \cdot \frac{2n\pi - \omega}{n} + \mathfrak{n}\right).
\end{aligned}$$

Ingrediuntur enim in hanc expressionem $2n$ constantes arbitrariae.

$$0 = y \pm \frac{2\,d^n y}{dx^n} + \frac{d^{2n} y}{dx^{2n}}$$

differentiale dx positum est constans, eius integrale invenire.

Solutio

equatio algebraica quae hinc formatur est:

$$0 = 1 \pm 2z^n + z^{2n} = (1 \pm z^n)^2,$$

um sit quadratum omnes eius factores erunt quadrati; pro signo s
ri haec forma

$$\left(1 - 2\,z \cos A \cdot \frac{2\,k - 1}{n}\,\pi + zz\right)^2$$

continet factores; pro signo inferiori autem haec forma

$$\left(1 - 2\,z \cos A \cdot \frac{2\,k}{n}\,\pi + zz\right)^2.$$

factoribus cognitis reperietur pro signo inferiori seu aequationis

$$0 = y - \frac{2\,d^n y}{dx^n} + \frac{d^{2n} y}{dx^{2n}}$$

le completum:

$$y = \left\{\begin{aligned} & Ae^x + Be^{x \cos A \cdot \frac{2}{n}\pi} \sin A \cdot \left(x \sin A \cdot \frac{2}{n}\pi + \mathfrak{B}\right) \\ & \qquad + Ce^{x \cos A \cdot \frac{4}{n}\pi} \sin A \cdot \left(x \sin A \cdot \frac{4}{n}\pi + \mathfrak{C}\right) \\ & \qquad + \text{etc.} \\ & + \alpha x e^x + \beta x e^{x \cos A \cdot \frac{2}{n}\pi} \sin A \cdot \left(x \sin A \cdot \frac{2}{n}\pi + \mathfrak{b}\right) \\ & \qquad + \gamma x e^{x \cos A \cdot \frac{4}{n}\pi} \sin A \cdot \left(x \sin A \cdot \frac{4}{n}\pi + \mathfrak{c}\right) \\ & \qquad + \text{etc.} \end{aligned}\right.$$

integrale erit

$$y = Ae^{x\cos A\cdot\frac{1}{n}\pi}\sin A\cdot\left(x\sin A\cdot\frac{1}{n}\pi + \mathfrak{A}\right.$$

$$+ Be^{x\cos A\cdot\frac{3}{n}\pi}\sin A\cdot\left(x\sin A\cdot\frac{3}{n}\pi + \mathfrak{B}\right.$$

$$+ Ce^{x\cos A\cdot\frac{5}{n}\pi}\sin A\cdot\left(x\sin A\cdot\frac{5}{n}\pi + \mathfrak{C}\right.$$

$$+ \text{etc.}$$

$$+ \alpha xe^{x\cos A\cdot\frac{1}{n}\pi}\sin A\cdot\left(x\sin A\cdot\frac{1}{n}\pi +\right.$$

$$+ \beta xe^{x\cos A\cdot\frac{3}{n}\pi}\sin A\cdot\left(x\sin A\cdot\frac{3}{n}\pi +\right.$$

$$+ \gamma xe^{x\cos A\cdot\frac{5}{n}\pi}\sin A\cdot\left(x\sin A\cdot\frac{5}{n}\pi +\right.$$

$$+ \text{etc.}$$

Q. E. I.

49. Ex his allatis exemplis iam abunde perspici
omnes aequationes differentiales cuiuscunque gradus,
tineantur in hac forma

$$0 = Ay + \frac{Bdy}{dx} + \frac{Cddy}{dx^2} + \frac{Dd^3y}{dx^3} + \frac{Ed^4y}{dx^4} +$$

denotantibus litteris A, B, C, D etc. coefficientes constan
integralia completa inveniri oporteat. Unica nimirum
resolutione aequationum algebraicarum in factores reales
trinomiales; quam autem in hoc negotio, quippe ab algebr
tanquam datam assumere possumus. At vero haec eaden
potest quoque in aequationibus huiusmodi, quarum term
grediuntur, dummodo aequationum algebraicarum, qu
omnes assignari queant radices. Hunc igitur usum unico ex

PROBLEMA XI

Si proposita fuerit ista aequatio differentialis in infinitum excurrens

$$0 = y - \frac{ddy}{2\,dx^2} + \frac{d^4y}{24\,dx^4} - \frac{d^6y}{720\,dx^6} + \frac{d^8y}{40320\,dx^8} - \text{etc.}$$

fferentiale dx positum est constans, eius integrale completum invenire.

Solutio

oto 1 pro y, et z^k pro differentiali cuiusvis gradus $\frac{d^ky}{dx^k}$, orietur ista in infinitum excurrens

$$0 = 1 - \frac{z^2}{1\cdot 2} + \frac{z^4}{1\cdot 2\cdot 3\cdot 4} - \frac{z^6}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6} + \frac{z^8}{1\cdot 2\cdots 8} - \text{etc.}$$

venit cum hac

$$0 = \cos A \cdot z.$$

go aequationis radices sunt omnes arcus circuli radii $= 1$, quorum vanescunt. Quocirca omnes possibiles valores ipsius z erunt sequentes

$$\pm\frac{\pi}{2},\quad \pm\frac{3\pi}{2},\quad \pm\frac{5\pi}{2},\quad \pm\frac{7\pi}{2},\quad \pm\frac{9\pi}{2}\ \text{etc.}$$

r radicibus ac proinde divisoribus simplicibus aequationis illius qui omnes sunt reales, aequationis differentialis propositae integrale m erit:

$$y = \alpha e^{\frac{\pi x}{2}} + \mathfrak{a} e^{\frac{-\pi x}{2}} + \beta e^{\frac{3\pi x}{2}} + \mathfrak{b} e^{\frac{-3\pi x}{2}} + \gamma e^{\frac{5\pi x}{2}} + \mathfrak{c} e^{\frac{-5\pi x}{2}}$$
$$+ \delta e^{\frac{7\pi x}{2}} + \mathfrak{d} e^{\frac{-7\pi x}{2}} + \text{etc. in infinitum.}$$

nus quisque terminus seorsim sumtus vel plures iuncti dabunt inte- rticulare aequationis differentialis propositae. Q. E. I.

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le talium aequationum exempla in *Institutionum calculi integralis* vol. II, § 1197 - 1202 , confer quoque notam p. 303 et praefationis p. IX. LEONHARDI EULERI *Opera omnia*, series I

H. D.

DE CONSTRUCTIONE AEQUATI

Commentatio 70 indicis ENESTROEMIANI

Commentarii academiae scientiarum Petropolitanae 9 (1737),

1. Quoties in resolutione problematum ad aequa pervenitur, ante omnia inquirendum est, an istae aequati admittant; perfectissime enim problema resolvi censendu structionem aequationis algebraicae deducitur. At si aeq sime evenit, in formam algebraicam nullo modo transmut quadraturis vel rectificationibus curvarum, quarum const problemata resolvenda uti oportet. Ad hoc vero efficiend aequatio solutionem problematis continens et primi tantu rentialis et praeterea separationem variabilium admittat, receptis atque iam satis cognitis uti velimus. Hoc enim ista defectu, ut earum ope neque aequationes differentiales neque differentiales primi gradus, quarum separatio non queant. Hanc ob rem nisi aequatio ad differentialem pri simulque separatio variabilium detegi potest, frustra per illa tio aequationis investigatur.

2. Dedi autem ego iam aliquoties specimina methodi[1]) o multo latius patentis, cuius ope non solum plures aequati separationem variabilium non admittentes construxi, sed differentiales secundi gradus, quae nequidem ad differenti reduci poterant. Initio quidem seriebus infinitis, in quas a

1) Vide p. 16, 20, 83 huius voluminis.

quisivi, qua ad easdem constructiones pertingere possem. In quo et
gotio operam non inutiliter collocavi; incidi enim in methodum aequati
dulares eruendi, quarum ope ad constructiones difficillimarum aequation
paratur. Methodum quidem hanc fusius iam exposui[1]), sed illius u
mium in construendis aequationibus illo tempore monstrare non vaca
terim tamen nuperrime dedi specimen illarum aequationum[2]), quae ope r
ationis ellipsis construi possunt. Nunc vero, quo usus huius methodi ple
spiciatur, casus nonnullos porvolvam speciales, ex quibus plurima
quationum constructiones consequantur. Principia autem ex dissertat
infinitis curvis eiusdem generis[1]), quam praecedente anno praelegi, pet

3. Cum igitur totum negotium ad inventionem aequationum me
ium recidat, sit $z = \int Pdx$, et P functio quaecunque ex x et a aliisque
ntibus conflata, in qua quidem integratione ipsius Pdx solum x ut varia
etetur. Quaeritur autem, si integrale $\int Pdx$ differentietur ponendo praet
am a variabile, quale differentiale sit proditurum. Inveniri igitur d
quatio differentialis vel primi, si fieri potest, vel altioris cuiusdam gra
qua a aeque insit tanquam variabilis ac x vel z. Huiusmodi ergo aequa
am cum HERMANNO modularem vocavi, tres continebit variabiles z, x
ae autem in aequationem duarum variabilium abibit, si vel ipsi z vel x de
natus vel ab a pendens valor tribuatur. Talis vero aequatio quamcun
buerit formam, et cuiuscunque sit gradus differentialis, semper ope ae
nis $z = \int Pdx$ construi poterit[3]). Nam si pro dato quoque ipsius a va
Pdx exhibeatur, quod per quadraturas fieri potest, et z vel x illi valori a
to aequale capiatur, determinabitur altera ipsarum z vel x per a, eius
e quantitas innotescit. Quocirca hac ratione pro dato alterius indetermin
lore alterius quantitas poterit reperiri, in quo ipsa aequationis cuiu
nstructio consistit.

4. Aequatio autem modularis erit vel differentialis primi gradus
cundi vel tertii vel altioris cuiusdam, prout functio P fuerit compar
quod dignoscendum et ipsam aequationem modularem inveniend

1) L. EULERI Commentationes 44 et 45 huius voluminis, p. 36 et p. 57. H.
2) L. EULERI Commentatio 52 voluminis I 20. Vide notam p. 16. H.
3) Cf. *Institutiones calculi integralis* vol. II, § 1017—1058; vide quoque notam p. 37. LEON
EULERI *Opera omnia*, series I, vol. 12, p. 221—245. H.

per da dividatur; quod prodit ponatur R. Porro simili mo
et per da dividendo orietur nova quantitas S, ex hacqu
Omnes ergo hae quantitates Q, R, S, T etc. ex data functio
His iam inventis positoque a iterum constante, si fueri

$$\int Qdx = \alpha \int Pdx + K,$$

ubi α utcunque datum esse potest per a et constantes, K vero
quamcunque ex a, x et constantibus conflatam; tum aeq
differentialis primi gradus, quae ex illa obtinetur, si loco
z et $\frac{dz - Pdx}{da}$ loco $\int Qdx$. Erit ergo aequatio modularis ha

$$\frac{dz - Pdx}{da} = \alpha z + K.$$

Haec vero quantitas K, quia quantitate constante quacun
minui, ita est accipienda, ut evanescat posito $x = 0$, si qu
Pdx ita accipi debeat, ut evanescat posito $x = 0$; quod
petuo est observandum. Loco K ergo semper scribi poter
quantitas, quae prodit, si in K ponatur $x = 0$.

5. Si $\int Qdx$ non pendeat a $\int Pdx$, ideoque aequatio h

$$\int Qdx = \alpha \int Pdx + K$$

inveniri nequeat, videndum est, num sit

$$\int Rdx = \alpha \int Qdx + \beta \int Pdx + K,$$

ubi iterum α et β per a et constantes, K vero per x, a et con
Si talis formae aequatio poterit formari, tum aequatio mo
tialis secundi gradus reperieturque per has formulas

$$\int Pdx = z, \int Qdx = \frac{dz - Pdx}{da},$$

$$\int Rdx = \frac{d\left(\frac{dz - Pdx}{da}\right) - Qdx}{da}.$$

sa inveniendi tam ex istis formulis tam ex sequentibus, quae sunt:

$$\int S dx = \frac{d\left(\frac{d\left(\frac{dz - Pdx}{da}\right) - Qdx}{da}\right) - Rdx}{da}$$

$\int T dx$ aequatur differentiali huius quantitatis ipso Sdx minuto et p
iso. Hocque modo ulterius est progrediendum, si aequatio modular
ferentialia altiorum graduum ascendat.

6. His praemissis praeceptis considerabo hanc aequationem special

$$z = \int e^{ax} X dx,$$

X functionem quamcunque ipsius x et constantium ab a non pend
nificet. Atque primo quidem investigabo, qualem valorem X habere de
aequatio modularis fiat tantum differentialis primi gradus, simulque c
di aequationes ope formulae

$$z = \int e^{ax} X dx$$

strui possint. Est vero e numerus, cuius logarithmus est unitas, atque
le ipsius $e^{ax} X dx$ ita sumi pono, ut evanescat posito $x = 0$. Cum igit
$= e^{ax} X$, et X ab a non pendeat, erit $e^{ax} X x da$ eius differentiale posito a
nte, ideoque

$$Q = e^{ax} X x.$$

o ergo aequatio modularis sit differentialis primi gradus, oportet sit

$$\int e^{ax} X x dx = a \int e^{ax} X dx + K - C.$$

namus $K = e^{ax} X p$ et sumantur differentialia posito a constante, habe

$$e^{ax} X x dx = a e^{ax} X dx + e^{ax} X dp + e^{ax} p dX + e^{ax} a X p dx$$

$$X x dx = a X dx + X dp + p dX + a X p dx.$$

le oritur

$$\frac{dX}{X} = \frac{x dx - a dx - dp - a p dx}{p},$$

1) Editio princeps: $\int e^{ax} X dx$ loco $\int e^{ax} X x dx$. Correxit F

at a utcunque ab a pendens effici potest.

7. Inventis autem hinc idoneis valoribus pro X e

$$dz - e^{ax} X dx = az da + (e^{ax} Xp - C)$$

Ponamus primo esse p constans $= m$, erit

$$\frac{dX}{X} = \frac{x dx - (a + ma) dx}{m},$$

fiatque

$$a + ma = b \text{ seu } a = b - ma,$$

ita ut b et m ab a non pendeant; erit

$$\frac{dX}{X} = \frac{x dx - b dx}{m} \text{ et } lX = \frac{x^2 - 2b}{2m}$$

atque

$$X = e^{\frac{x^2 - 2bx}{2m}};$$

constans vero C erit $= m$. Quamobrem ex aequatione

$$z = \int e^{\frac{x^2 - 2bx + 2max}{2m}} dx$$

oritur ista aequatio modularis

$$dz = (b - ma) z da - m da + e^{\frac{x^2 - 2bx + 2max}{2m}} ($$

Haec ergo aequatio, cuicunque functioni ipsius a quant ut duae tantum variabiles z et a supersint, semper quidem aliunde iam patet, quia altera variabilis z unic At si ipsi z datus per a et constantes valor tribuatur, h variabiles a et x tantum, quae consueto more minus trac tamen hoc modo construi poterit: pro quovis ipsius a va cuius applicata abscissae x respondens sit

$$= e^{\frac{x^2 - 2bx + 2max}{2m}}$$

in hacque curva sumatur area aequalis eidem ipsius aequalis, erit abscissa hoc modo determinata verus va

$$p = \beta + \gamma x,$$

$$\frac{dX}{X} = \frac{xdx - adx - \gamma dx - \beta a dx - \gamma a x dx}{\beta + \gamma x},$$

ressio, quo a ex ea excedat, ponatur

$$\frac{dX}{X} = \frac{fxdx - gdx}{mx + n},$$

m et n non involvant a, erit

$$\beta = \frac{n}{f + ma}, \gamma = \frac{m}{f + ma}$$

$$a = \frac{g - m - na}{f + ma} \text{ atque } p = \frac{n + mx}{f + ma}.$$

ur

$$lX = \frac{fx}{m} - \frac{fn + gm}{m^2} l(mx + n)$$

$$\text{atque } X = e^{\frac{fx}{m}}(mx + n)^{\frac{-fn - gm}{m^2}}$$

$$K = e^{ax + \frac{fx}{m}}(mx + n)^{\frac{m^2 - fn - gm}{m^2}} : (f + ma),$$

$$C = \frac{n^{\frac{m^2 - fn - gm}{m^2}}}{f + ma}.$$

$f = 0$, quod sine detrimento universalitatis fieri potest, erit

$$z = \int e^{ax}(mx + n)^{\frac{-g}{m}} dx;$$

uens oriotur aequatio modularis

$$\frac{(g - m - na) zda}{ma} + \frac{e^{ax}(mx + n)^{\frac{-g}{m}}(madx + nda + mxda)}{ma} - \frac{n^{\frac{m - g}{m}} da}{ma}.$$

omodocunque z per a ita ut sit

habebitur constructio huius aequationis

$$Ada = e^{ax}(mx + n)^{\frac{-g}{m}}(madx + nda + \ldots$$

quae quidem facta substitutione $x = \frac{y - na}{ma}$ facile se…

9. Cum igitur hae aequationes, quae ex aequation…
rentialibus primi gradus eliciuntur, receptas regula…
superent, progrediendum est ad aequationes modulare…
gradus. Retinebo vero priorem formam $z = \int e^{ax}Xdx$ et …
functionem ipsius x esse oporteat X, quo aequatio m…
differentialia ascendat. Erit vero

$$P = e^{ax}X,\ Q = e^{ax}Xx \ \text{ et } \ R = e^{a\ldots}$$

quare pono

$$\int e^{ax}Xx^2dx = \alpha\int e^{ax}Xxdx + \beta\int e^{ax}Xdx \ldots$$

Sumatur

$$K = e^{ax}Xp,$$

habebitur sumtis differentialibus

$$Xx^2dx = \alpha Xxdx + \beta Xdx + Xdp + pdX \ldots$$

unde fit

$$\frac{dX}{X} = \frac{x^2dx - \alpha xdx - \beta dx - dp - a\ldots}{p}$$

Ponatur

$$p = \frac{(x-\gamma)(x-\delta)}{a},$$

erit

$$\frac{dX}{X} = \frac{-dp}{p} + \frac{a(\gamma + \delta - \alpha)xdx - a(\gamma\delta \ldots}{(x-\gamma)(x-\delta)}$$

Sit

$$a\gamma + a\delta - a\alpha = f \ \text{ seu } \ \alpha = \gamma + \delta - \frac{f}{a} \ \text{ et}$$

existentibus γ, δ et f, g quantitatibus ab α non pende…

$$\frac{dX}{X} = \frac{-dp}{p} + \frac{fxdx - gdx}{(x-\gamma)(x-\delta)}$$

atque

$$X = c\,(x-\gamma)^{\frac{\gamma f - g - \gamma + \delta}{\gamma - \delta}} (x-\delta)^{\frac{\delta f - g - \delta + \gamma}{\delta - \gamma}}.$$

10. Ponatur

$$\frac{\gamma f - g - \gamma + \delta}{\gamma - \delta} = \lambda \text{ et } \frac{\delta f - g - \delta + \gamma}{\delta - \gamma} = \mu,$$

erit

$$f = \lambda + \mu + 2 \text{ et } g = \gamma\mu + \delta\lambda + \gamma + \delta.$$

Hinc erit

$$X = c\,(x-\gamma)^\lambda (x-\delta)^\mu, \quad \alpha = \gamma + \delta - \frac{\lambda + \mu + 2}{a}$$

et

$$\beta = \frac{\gamma\mu + \delta\lambda + \gamma + \delta}{a} - \gamma\delta,$$

atque

$$K = \frac{c e^{ax} (x-\gamma)^{\lambda+1} (x-\delta)^{\mu+1}}{a}$$

et

$$C = \frac{c\,(-\gamma)^{\lambda+1} (-\delta)^{\mu+1}}{a}.$$

Quocirca fiet

$$z = \int e^{ax} (x-\gamma)^\lambda (x-\delta)^\mu c\,dx,$$

quae dabit sequentem aequationem modularem

$$\begin{aligned} d\left(\frac{dz - e^{ax}(x-\gamma)^\lambda (x-\delta)^\mu c\,dx}{da}\right) &= e^{ax}(x-\gamma)^\lambda (x-\delta)^\mu c x\,dx \\ &+ (\gamma + \delta)\,dz - \frac{(\lambda + \mu + 2)\,dz}{a} \\ &- \left(\gamma + \delta - \frac{\lambda + \mu + 2}{a}\right) e^{ax}(x-\gamma)^\lambda (x-\delta)^\mu c\,dx \\ &+ \frac{(\gamma\mu + \delta\lambda + \gamma + \delta)\,z\,da}{a} - \gamma\delta z\,da \\ &+ \frac{e^{ax}(x-\gamma)^{\lambda+1} (x-\delta)^{\mu+1} c\,da}{a} - \frac{(-\gamma)^{\lambda+1} (-\delta)^{\mu+1} c\,da}{a}. \end{aligned}$$

Sive quod eodem redit

$$z = \int e^{ax} (\varepsilon x + \eta)^\lambda (\zeta x + \theta)^\mu dx$$

1) Cf. *Institutiones calculi integralis* vol. II, § 1036—1039. Vide notam p. 151.

$$d\left(\frac{dz - e^{ax}(\varepsilon x + \eta)^\lambda (\zeta x + \theta)^\mu dx}{da}\right) = e^{ax}(\varepsilon x + \eta)^\lambda (\zeta$$

$$-\left(\frac{\eta}{\varepsilon} + \frac{\theta}{\zeta} + \frac{\lambda + \mu + 2}{a}\right)\left(dz - e^{ax}(\varepsilon x + \eta)^\lambda (\zeta x + \theta\right.$$

$$-\left(\frac{\eta(\mu+1)}{\varepsilon a} + \frac{\theta(\lambda+1)}{\zeta a} + \frac{\eta\theta}{\varepsilon\zeta}\right) z da$$

$$+ e^{ax}(\varepsilon x + \eta)^{\lambda+1}(\zeta x + \theta)^{\mu+1}\frac{da}{\varepsilon\zeta a} - \frac{\eta^{\lambda+1}\theta^{\mu+1} da}{\varepsilon\zeta a},$$

in qua litterae $\varepsilon, \zeta, \eta, \lambda, \mu$ denotant quantitates constantes

11. Tribuatur ipsi x valor vel constans vel ab a quom et sumto da constante loco omnium terminorum, in quibus Ada denotante A functionem resultantem ipsius a et cons abibit aequatio modularis in sequentem aequationem dua z et a involventem:

$$\frac{ddz}{da} + \left(\frac{\eta}{\varepsilon} + \frac{\theta}{\zeta} + \frac{\lambda + \mu + 2}{a}\right) dz + \left(\frac{\eta(\mu+1)}{\varepsilon a} + \frac{\theta(\lambda+1)}{\zeta a} + \right.$$

seu

$$\frac{ddz}{da} + \left(b + \frac{c}{a}\right) dz + \left(f + \frac{g}{a}\right) z da = A da$$

positis

$$\frac{\eta}{\varepsilon} + \frac{\theta}{\zeta} = b, \quad \lambda + \mu + 2 = c, \quad \frac{\eta\theta}{\varepsilon\zeta} = f$$

et

$$\frac{\eta(\mu+1)}{\varepsilon} + \frac{\theta(\lambda+1)}{\zeta} = g.$$

Haec ergo aequatio differentio-differentialis ope aequatio

$$z = \int e^{ax}(\varepsilon x + \eta)^\lambda (\zeta x + \theta)^\mu dx$$

poterit construi. Simili modo si ipsi z tribuatur valor v pendens, aequatio modularis abibit in aequationem diffe inter x et a multo magis implicatam, cuius nihilominu exhiberi.

12. Quo autem obtineamus aequationes differentiale hoc modo construi queant, oportet, ut aequationes ita er

nisi plures litterarum arbitrariarum definire velimus. Assumo ergo aeq
fundamentalem magis compositam hanc

$$z = E\int e^{ax}(\eta + \varepsilon x)^{\lambda}(\theta + \zeta x)^{\mu}dx + F\int e^{-ax}(\eta - \varepsilon x)^{\lambda}(\theta - \zeta x)^{\mu}d$$

ubi E, F, ε, ζ, η, θ, λ, μ sint quantitates constantes ab a non pendent
vero ut ante

$$b = \frac{\theta}{\zeta} + \frac{\eta}{\varepsilon}, \quad c = \lambda + \mu + 2, \quad f = \frac{\eta\theta}{\varepsilon\zeta}$$

et

$$g = \frac{\eta(\mu + 1)}{\varepsilon} + \frac{\theta(\lambda + 1)}{\zeta},$$

invenietur ex hac aequatione sequens modularis:

$$d\left(\frac{dz - Ee^{ax}(\eta + \varepsilon x)^{\lambda}(\theta + \zeta x)^{\mu}dx - Fe^{-ax}(\eta - \varepsilon x)^{\lambda}(\theta - \zeta x)^{\mu}dx}{da}\right)$$

$$= Ee^{ax}(\eta + \varepsilon x)^{\lambda}(\theta + \zeta x)^{\mu}xdx - Fe^{-ax}(\eta - \varepsilon x)^{\lambda}(\theta - \zeta x)^{\mu}xdx$$

$$-\left(b + \frac{c}{a}\right)(dz - Ee^{ax}(\eta + \varepsilon x)^{\lambda}(\theta + \zeta x)^{\mu}dx - Fe^{-ax}(\eta - \varepsilon x)^{\lambda}(\theta -$$

$$-\left(f + \frac{g}{a}\right)zda + \frac{Ee^{ax}(\eta + \varepsilon x)^{\lambda+1}(\theta + \zeta x)^{\mu+1}da}{\varepsilon\zeta a}$$

$$-\frac{Fe^{-ax}(\eta - \varepsilon x)^{\lambda+1}(\theta - \zeta x)^{\mu+1}da}{\varepsilon\zeta a} - \frac{(E - F)\eta^{\lambda+1}\theta^{\mu+1}da}{\varepsilon\zeta a}.$$

13. Quo nunc talis valor pro x substituendus inveniatur,
termini praeter eos in quibus inest z evanescant, facio $E = F = 1$, q
nus ultimus evanescat. Deinde pono

$$\frac{\eta}{\varepsilon} + \frac{\theta}{\zeta} = 0 \quad \text{seu} \quad b = 0, \quad \text{atque facio} \quad x = \frac{-\eta}{\varepsilon},$$

ut ambo termini penultimi evanescant, ad quod quidem requiritur
et $\mu + 1$ sint numeri affirmativi. Quia itaque x constantem habet
omnes termini in quibus inest dx evanescent. Fiat brevitatis gratia

$$\varepsilon = -1, \quad \zeta = 1, \quad \text{et} \quad \eta = \theta = h,$$

erit

$$b = 0, \quad c = \lambda + \mu + 2, \quad f = -h^2 \quad \text{et} \quad g = \lambda h - \mu h = h(\lambda -$$

In qua si sumatur $x = h$ et a tanquam variabilis tractetur, aequatio inter z et a, si da constans ponatur:

$$\frac{ddz}{da} + \frac{cdz}{a} + \left(f + \frac{g}{a}\right)zda = 0,$$

quae in aequationem differentialem primi gradus transi $z = e^{\int tda}$, prodibit enim

$$dt + t^2da + \frac{ctda}{a} + \left(f + \frac{g}{a}\right)da = 0.$$

Ponatur

$$ta^c = y \quad \text{seu} \quad t = a^{-c}y,$$

habebitur

$$dy + \frac{y^2da}{a^c} + (fa^c + ga^{c-1})da = 0.$$

Fiat porro

$$a^{1-c} = u,$$

erit

$$\frac{da}{a^c} = \frac{du}{1-c}$$

ideoque

$$dy + \frac{y^2du}{1-c} + \frac{f}{1-c}u^{\frac{2c}{1-c}}du + \frac{g}{1-c}u^{\frac{2c-1}{1-c}}du = ($$

seu

$$(\lambda + \mu + 1)dy = y^2du - h^2u^{\frac{-2\lambda-2\mu-4}{\lambda+\mu+1}}du + h(\lambda - \mu)u^{\frac{-2}{\lambda}}$$

Ponatur

$$\lambda + \mu = m, \quad \lambda - \mu = n,$$

habebitur ista aequatio

$$(m+1)dy = y^2du - h^2u^{\frac{-2m-4}{m+1}}du + nhu^{\frac{-2m-3}{m+1}}d$$

quae construi potest ex aequatione

$$z = \int e^{ax}(h-x)^{\frac{m+n}{2}}(h+x)^{\frac{m-n}{2}}dx + \int e^{-ax}(h+x)^{\frac{m+n}{2}}(h -$$

Nam si post integrationem ita institutam, ut posito $x = 0$ z eva $x = h$ et pro a substituatur $u^{\frac{-1}{m+1}}$, habebitur functio ipsius u

est verus valor ipsius y in aequatione inventa. Notandum vero est
$m - n$ numeros affirmativos esse debere.

14. Si tam $\frac{m+n}{2}$ quam $\frac{m-n}{2}$ fuerint numeri integri affirmativi, tu
ius z per integrationem poterit exhiberi et proinde valor ipsius V
signari. His igitur casibus aequatio proposita

$$(m+1)dy = y^2du - h^2u^{\frac{-2m-4}{m+1}}du + nhu^{\frac{-2m-3}{m+1}}du$$

ore consueto poterit integrari eiusque integrale exhiberi. Ponatur er

$$m = i + k, \text{ et } n = i - k$$

notantibus i et k numeris integris affirmativis, et habebimus hanc
nem

$$(1 + i + k)dy = y^2du - h^2u^{\frac{-2i-2k-4}{i+k+1}}du + (i-k)hu^{\frac{-2i-2k-3}{i+k+1}}du;$$

ae non solum modo supra exposito construi, sed etiam consueto mor
i et integrari poterit. Nam in aequatione

$$z = \int e^{ax}(h-x)^i(h+x)^kdx + \int e^{-ax}(h+x)^i(h-x)^kdx$$

st integrationem, quae actu succedet, ita institutam, ut posito
anescat z, ponatur $x = h$ et pro a substituatur hic valor $u^{\frac{-1}{i+k+}}$
to z aequabitur functioni cuidam ipsius u, quae sit V; invento vero

$$y = \frac{-(i+k+1)dV}{Vdu}.$$

fiat insuper $k = i$, prodibit aequatio a Com. RICCATO quondam pr

$$(1 + 2i)dy = y^2du - h^2u^{\frac{-4i-4}{2i+1}}du,$$

ius adeo constructio[1]) universalis est exhibita.

1) Si fiat $i = k$, quamquam i non integrum sit, haec constructio in constructionem C
ionum 11 et 31 huius voluminis coalescet. Cf. formulam ipsius z superius scriptam cum
ae in Commentatione 31, § 17, huius voluminis p. 34, scribitur.

DE AEQUATIONIBUS DIFFERENT
QUAE CERTIS TANTUM CASIBUS INT
ADMITTUNT

Commentatio 95 indicis Enestroemiani
Commentarii academiae scientiarum Petropolitanae 10 (1738),

1. Cum ad aequationes differentiales, quae generaliter
methodis adhuc usitatis pervenitur, non parum augmen
censenda est, si casus saltem particulares assignentur
locum inveniat. Dum enim integratio casuum ab integrati
tionis non pendet, eo magis erit abscondita atque inventu
per generaliores integrandi methodos perfici poterit. Tali
complures annos a Comite Riccato[1]) est producta, atque a
geometris multum agitata, ex qua satis perspicere licet, q
integrabiles per alias methodos tractarentur, nisi redu
casuum ad simpliciores uti vellemus. Casus scilicet isti
inventi, ut idonea facta substitutione casus simplicissimu
promtu est, in alium transmutetur eadem forma general
denuo in alium et ita porro in infinitum, quo facto hor
integratio ex simplicissimo consequitur.

2. Proponam hic autem aliam methodum latius
solum in aequatione illa Riccatiana, sed etiam in plu
integrationem pariter respuentibus, casus integrabiles erui

1) Vide notam 1 p. 17 huius voluminis.

tam integralia finitis aequationibus exprimerent. Sed cum quae
[a]quatio plurimis modis per seriem integrari possit, difficillimum plerum
in eiusmodi seriem incidere, quae certis casibus abrumpatur; ita aequa
n illam RICCATIANAM per varias substitutiones in aliam formam transmu
ortet, antequam integratio per seriem eiusmodi absolvi queat, quae casi
egrabilibus abrumpatur.

3. Talis autem praeparatio, quae ad seriem idoneam manuducat,
do fieri nequit, nisi ut aequatio proposita in aequationem differentia
undi vel altioris cuiusdam gradus transmutetur, in qua altera varia
ique unam tantum obtineat dimensionem; huiusmodi enim aequatio fa
commode per seriem integrari potest. At hoc solum non sufficit ad pro
um nostrum; series enim praeterea haec ita debet esse comparata, ut ce
ibus abrumpi queat, quod evenit, si facto coefficiente uniuscuius
mini = 0 sequentium terminorum omnium coefficientes simul evanesc
m igitur haec praeparatio tantis laboret difficultatibus, expediet negotiu
steriori aggredi, atque primo aequationem differentialem secundi gra
eralissimam contemplari, cuius integratio per seriem absoluta hac gau
erogativa, ut infinitis casibus fiat finita; quibus adeo casibus aequ
umta integrari poterit. Hoc facto aequationem istam differentialem secu
dus ad differentialem primi gradus reducam, eamque in varias for
nsmutabo, quo plurimas imo infinitas obtineam aequationes differenti
mi gradus, quae iisdem casibus sint integrabiles. Hinc autem non so
spicuum erit, aequationes inventas illis casibus esse integrabiles, sed re
diendo etiam ipsa aequatio integralis assignari poterit.

4. Huiusmodi autem aequatio differentialis secundi gradus, q
uisitis illis satisfaciat, atque latissime pateat, est haec[1]):

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1) Cf. Commentationem 284 huius voluminis. Vide *Institutiones calculi integralis* vo
20—991, 997—1007, 1014, 1033—1036, 1069—1080. Vide porro L. EULERI Commentationem
sideratio aequationis differentio-differentialis

$$(a + bx)\,ddz + (c + ex)\frac{dx\,dz}{x} + (f + gx)\frac{z\,dx^2}{xx} = 0.$$

vi comment. acad. sc. Petrop. 17, 1773, p. 129. *LEONHARDI EULERI Opera omnia*, series I, vol. 1
H.

$$(a + bx^n) x^2 ddv + (c + fx^n) xdxdv + (g + hx^n) vdx^2$$

in qua variabilis x elementum dx positum est constans. Ex hac tione valor ipsius v duplici modo per seriem definiri potest, qu si ponatur

$$v = Ax^m + Bx^{m+n} + Cx^{m+2n} + Dx^{m+3n} + Ex^{m+4n} +$$

Hinc enim valoribus loco v, dv et ddv substitutis, et termin factis $= 0$, sequentes prodibunt coefficientium A, B, C, D etc. determinationes. Primo enim debet esse

$$g + cm + am(m-1) = 0,$$

unde ne ad irrationalia perveniamus, m potius tamquam nume spectemus ex eoque g determinemus, eritque

$$g = -cm - am(m-1).$$

Deinde vero habebimus hoc valore loco g ubique substituto

$$B = \frac{-A(h + fm + bm(m-1))}{cn + an(2m + n - 1)}$$

$$C = \frac{-B(h + f(m+n) + b(m+n)(m+n-1))}{2cn + 2an(2m + 2n - 1)}$$

$$D = \frac{-C(h + f(m+2n) + b(m+2n)(m+2n-1))}{3cn + 3an(2m + 3n - 1)}$$

$$E = \frac{-D(h + f(m+3n) + b(m+3n)(m+3n-1))}{4cn + 4an(2m + 4n - 1)}$$

etc.

Erit ergo A quantitas constans arbitraria, a qua sequentes coeffi pendent.

5. Ex his coefficientium valoribus inventis intelligitur, si ciens evanuerit, sequentes omnes simul evanescere, ita, ut his ipsius v fiat finitus, atque ideireo aequatio assumta

$$(a + bx^n)x^2ddv + (c + fx^n)xdxdv + (g + hx^n)vdx^2 =$$

integrationem admittat. Si enim fuerit

$$h + fm + bm(m-1) = 0,$$

tum erit $v = Ax^m$; sin autem sit

$$h + f(m+n) + b(m+n)(m+n-1) = 0,$$

tum erit

$$v = Ax^m + Bx^{m+n},$$

atque si

$$h + f(m+2n) + b(m+2n)(m+2n-1) = 0,$$

erit

$$v = Ax^m + Bx^{m+n} + Cx^{m+2n}.$$

Semper igitur aequatio proposita integrationem admittet, quoties fue

$$h + f(m+in) + b(m+in)(m+in-1) = 0,$$

seu

$$h = -f(m+in) - b(m+in)(m+in-1)$$

denotante i numerum quemcumque integrum affirmativum cyphra non Interim tamen ii excipiendi sunt casus quibus denominatores evane ista integratio non succedit, si fuerit

$$c = -a(2m + (i+1)n - 1),$$

si quidem hoc casu i minor fuerit quam illo.

6. Alter modus ex nostra aequatione valorem ipsius v per serie in hoc constat, ut ponatur

$$v = Ax^k + Bx^{k-n} + Cx^{k-2n} + Dx^{k-3n} + Ex^{k-4n} + \text{etc.}$$

Hinc enim pro v, dv et ddv debitis valoribus surrogandis reperietur

$$h + fk + bk(k-1) = 0,$$

quare ponamus

$$h = -fk - bk(k-1).$$

Porro vero erit

$$B = \frac{A(g + ck + ak(k-1))}{nf + nb(2k-n-1)}$$

$$C = \frac{B(g + c(k-n) + a(k-n)(k-n-1))}{2fn + 2bn(2k-2n-1)}$$

$$D = \frac{C(g + c(k-2n) + a(k-2n)(k-2n-1))}{3fn + 3bn(2k-3n-1)}$$

$$E = \frac{D(g + c(k-3n) + a(k-3n)(k-3n-1))}{4fn + 4bn(2k-4n-1)}$$

etc.

denotante ut ante i numerum quemcunque integrum
aequatio proposita erit integrabilis. Namque

si $i = 0$ erit $v = Ax^k$,

si $i = 1$ erit $v = Ax^k + Bx^{k-n}$,

si $i = 2$ erit $v = Ax^k + Bx^{k-n} + Cx^{k-2n}$

et ita porro.

7. Aequatio ergo nostra generalis

$$(a + bx^n)x^2 ddv + (c + fx^n)xdxdv + (g + hx^n$$

in qua est

$$g = -cm - am(m-1) \quad \text{atque} \quad h = -fk -$$

quibus definitionibus nulla vis amplitudini aequationis
arbitrariarum quantitatum g et h duae novae arbitrariae
haec, inquam, aequatio integrationem admittit, quoties fi

vel $f = \dfrac{(m + in)(m + in - 1) - k(k-1)}{k - m - in} b = (1 - k - m -$

vel $c = \dfrac{(k - in)(k - in - 1) - m(m-1)}{m - k + in} a = (1 - k - m +$

Duplici ergo modo infiniti casus assignari possunt, quibu
integrabilis existit; atque insuper his singulis casibus ipsa
ipsius v per x algebraice exprimi poterunt, quaerendo v
B, C, D etc.; quippe quorum numerus istis casibus fiet fi

8. Quamvis autem hoc modo casuum erutorum
inveniantur, tamen non est putandum haec integralia ae
tiones differentiales ex quibus sunt ortae. Quemadmo
ipsius dx non solum est x sed etiam $x + a$, ita haec integ
hoc modo inveniuntur, sunt tantum casus particulares p
qui oriuntur, si constans quaepiam arbitraria vel nihilo
ponatur. Interim tamen in his omnibus casibus, quib

$$Pddv + Qdxdv + Rvdx^2 = 0,$$

, Q, R sint functiones quaecunque ipsius x, cuius iam inventum sit in
particulare per huiusmodi viam, scilicet $v = X$, hoc est functioni cuid
x. Iam ad aequationem integralem completam eruendam pono

$$v = Xz, \quad \text{erit} \quad dv = zdX + Xdz$$

$$\text{atque} \quad ddv = zddX + 2\,dXdz + Xddz,$$

s substitutis aequatio proposita abibit in hanc

$$\begin{aligned} &+ PzddX + 2\,PdXdz + PXddz = 0; \\ &+ QzdXdx + QXdxdz \\ &+ RzXdx^2 \end{aligned}$$

m X sit valor, qui pro v substitutus satisfacit, erit

$$PddX + QdXdx + RXdx^2 = 0.$$

rea deletis his terminis restabit

$$2\,PdXdz + QXdxdz + PXddz = 0$$

$$\frac{2\,dX}{X} + \frac{Qdx}{P} + \frac{ddz}{dz} = 0;$$

a cum P et Q sint functiones ipsius x, ponatur

$$\int \frac{Qdx}{P} = S$$

o integrando

$$X^2dz = Ce^{-S}dx \quad \text{atque} \quad z = C\int \frac{e^{-S}dx}{X^2}$$

ante e numerum cuius logarithmus hyperbolicus est 1. Aequationis c

$$Pddv + Qdxdv + Rvdx^2 = 0,$$

tisfacit $v = X$, completum integrale erit

$$v = CX\int \frac{e^{-\int \frac{Qdx}{P}}}{X^2}\,dx.$$

integrationem admittat, atque simul etiam horum casu
inveniri queant, inquiramus in aequationes differentia
ex ista resultent, atque ideo iisdem casibus integrab
autem proposita facile in aequationem differentialem
mutatur ponendo

$$v = e^{\int z dx}, \text{ ita ut sit } z = \frac{dv}{vdx}\,.$$

Unde cognito valore ipsius v, simul valor ipsius z innotes

$$dv = e^{\int z dx}\, zdx \text{ et } ddv = e^{\int z dx}\,(dxdz +$$

quibus valoribus substitutis aequatio nostra transibit i

$$(a + bx^n)x^2dz + (c + fx^n)xzdx + (a + bx^n)x^2z^2dx +$$

Haec ergo aequatio differentialis primi gradus factis

$$g = -cm - am\,(m-1) \quad \text{et} \quad h = -fk -$$

semper est integrabilis, si fuerit

$$\text{vel } f = \frac{(m+in)(m+in-1)-k(k-1)}{k-m-in}\,b = (1-k-m$$

$$\text{vel } c = \frac{(k-in)(k-in-1)-m(m-1)}{m-k+in}\,a = (1-k-m$$

quibus casibus etiam ex valore ipsius v invento valor i
quam incompletus ope aequationis $z = \dfrac{dv}{vdx}$ invenietur.

10. Quo autem clarius appareat, quales aequatio
generali contineantur, in aliam formam aequationem in
in qua tres tantum insint termini huius formae

$$Pdz + Qz^2dx + Rdx = 0$$

denotantibus P, Q et R functiones ipsius x. Haec
modis fieri potest, quorum primus est, si ponatur $z =$
ipsius x etiamnum incognita. Facta ergo hac substitutio

$$(c+fx^n)T\,dx + (a+bx^n)x\,dT = 0,$$

...minus, qui y continet, evanescat; habebitur ergo

$$\frac{(c+fx^n)dx}{(a+bx^n)x} + \frac{dT}{T} = 0,$$

...orem ipsius T erui oportet. Reducetur autem haec aequatio ad istam

$$\frac{c\,dx}{ax} + \frac{(af-bc)x^{n-1}dx}{a(a+bx^n)} + \frac{dT}{T} = 0,$$

...ograle est

$$\frac{c}{a}lx + \frac{af-bc}{abn}l(a+bx^n) + lT = C$$

$$T = \frac{(a+bx^n)^{\frac{bc-af}{abn}}}{x^{\frac{c}{a}}}.$$

...rgo

$$z = \frac{(a+bx^n)^{\frac{bc-af}{abn}}y}{x^{\frac{c}{a}}}$$

... nostra abibit in hanc

$$dy + \frac{(a+bx^n)^{\frac{bc-af}{abn}}y^2dx}{x^{\frac{c}{a}}} + \frac{(g+hx^n)x^{\frac{c}{a}-2}dx}{(a+bx^n)^{\frac{bc-af}{abn}+1}} = 0$$

...pterea iisdem casibus, quibus superiores aequationes, integrationem

... Hinc iam specialiores formemus aequationes ponendo primo ...ut sit

$$dy + x^{\frac{-c}{a}}y^2dx + \frac{(g+hx^n)x^{\frac{c}{a}-2}dx}{a+bx^n} = 0.$$

...porro

$$x^{\frac{a-c}{a}} = t \quad \text{seu} \quad x = t^{\frac{a}{a-c}}$$

...r

Haec ergo aequatio, si fuerit

$$g = -cm - am(m-1) \quad \text{et} \quad h = -\frac{b}{a}(ck + ak(k -$$

semper integrationem admittet, quoties erit

$$\text{vel} \quad c = (1 - k - m - in)a \quad \text{vel} \quad c = (1 - k - m +$$

hoc est quoties erit

$$\frac{c + a(k + m - 1)}{an}$$

numerus integer sive affirmativus sive negativus.

12. Si insuper fuerit $c = 0$, habebitur loco g et h actu valoribus

$$dy + y^2 dt = \frac{(am(m-1) + bk(k-1)t^n)dt}{(a + bt^n)tt}$$

quae aequatio integrabilis erit, quoties fuerit

$$\text{vel} \quad \frac{1 - k - m}{n} \quad \text{vel} \quad \frac{k + m - 1}{n}$$

numerus integer affirmativus; hoc est quoties fuerit $\frac{k + m - 1}{n}$ sive affirmativus sive negativus. Haec ergo aequatio

$$dy + y^2 dt = \frac{am(m-1)dt}{(a + bt^n)tt}$$

integrabilis erit, si fuerit vel $\frac{m-1}{n}$ vel $\frac{m}{n}$ numerus integer sive a negativus. Atque haec aequatio

$$dy + y^2 dt = \frac{bk(k-1)t^n dt}{(a + bt^n)tt}$$

integrabilis erit, si vel $\frac{k-1}{n}$ vel $\frac{k}{n}$ fuerit numerus integer sive a negativus.

$$dy + \frac{\quad}{x} = \frac{\quad}{(a + bx^n)x}$$

emper integrationem admittet, quoties fuerit $\frac{k+m}{n}$ numerus integer siv
ativus sive negativus. Quare haec aequatio

$$dy + \frac{y^2dx}{x} = \frac{m^2adx}{(a + bx^n)x}$$

abilis erit, quoties $\frac{m}{n}$ fuerit numerus integer; haec vero aequatio

$$dy + \frac{y^2dx}{x} = \frac{k^2bx^{n-1}dx}{a + bx^n},$$

s $\frac{k}{n}$ fuerit numerus integer.

4. Resumamus aequationem generalem

$$dy + \frac{(a + bx^n)^{\frac{be-af}{abn}} y^2dx}{x^{\frac{e}{a}}} + \frac{(g + hx^n)x^{\frac{e}{a}-2}dx}{(a + bx^n)^{\frac{be-af}{abn}+1}} = 0,$$

amus

$$e = -a(n-1), \text{ fiatque } (a + bx^n)^{\frac{b-f}{bn}} = t,$$

$$x^n = \frac{t^{\frac{bn}{b-f}} - a}{b};$$

it ista aequatio

$$dy + \frac{y^2dt}{b-f} + \frac{b\left(bg - ah + ht^{\frac{bn}{b-f}}\right)t^{\frac{bn}{b-f}-2}dt}{(b-f)\left(t^{\frac{bn}{b-f}} - a\right)^3} = 0,$$

est

$$g = am(n-m) \text{ et } h = -fk - bk(k-1).$$

vero aequatio toties integrabilis evadit, quoties fuerit

vel $\frac{k+m-n}{n}$ numerus integer affirmativus seu i

vel $\frac{f + b(m+k-1)}{bn}$ numerus integer negativus.

quae semper integrationem admittet, dummodo $\frac{k+m}{n}$ fue
sive affirmativus sive negativus. Hinc posito $k=n$, ista a

$$dy+\frac{y^2dt}{nb}+\frac{abm(n-m)dt}{nt(t-a)^2}=0$$

integrationem admittet, si fuerit $\frac{m}{n}$ numerus integer. At
aequatio

$$dy+\frac{y^2dt}{nb}+\frac{bk(n-k)dt}{nt(t-a)}=0$$

integrabilis erit, quando fuerit $\frac{k}{n}$ numerus integer siv
negativus.

15. Revertamur ad aequationem primitivam inter x

$(a+bx^n)x^2dz+(c+fx^n)xzdx+(a+bx^n)x^2z^2dx+(g$

quae posito

$$g=-cm-am(m-1) \text{ et } h=-fk-bk$$

integrabilis est, si fuerit

$$\text{vel} \quad f=(1-k-m-in)b \quad \text{vel} \quad c=(1-k-$$

Alio autem modo eam transformemus in aequationem trib
constantem. Ponamus scilicet

$$z=Ty+S,$$

denotantibus T et S functionibus ipsius x; erit

$$dz=Tdy+ydT+dS,$$

his substitutis prodibit ista aequatio

$$\begin{aligned}(a+bx^n)Tx^2dy+(a+bx^n)x^2ydT+(a+bx^n)x^2T^2y^2dx&+\\ +(c+fx^n)Txydx\qquad\qquad\qquad\qquad&+\\ +2(a+bx^n)x^2TSydx\qquad\qquad\qquad&+\\ &+\end{aligned}$$

$$\frac{dT}{T} + 2\,Sdx + \frac{(c + fx^n)dx}{(a + bx^n)x} = 0.$$

us ante omnia $T = x^p$, quo post divisionem per $(a + bx^n)Txx$ coefficie
y^2dx fiat simplex potestas ipsius x; erit

$$\frac{p}{x} + 2\,S + \frac{c + fx^n}{x(a + bx^n)} = 0$$

$$S = \frac{-c - ap - (f + bp)x^n}{2\,x(a + bx^n)}.$$

et

$$\frac{a(c + ap)dx - a(n - 1)(f + bp)x^n dx + b(f + bp)x^{2n} dx + b(n+1)(c + ap)x^n a}{2\,xx(a + bx^n)^2}$$

his valoribus substitutis obtinebitur ista aequatio

$$(a + bx^n)\,x^{p+2}dy + (a + bx^n)\,x^{2p+2}y^2dx + \frac{p(p + 2)(a + bx^n)dx}{4}$$

$$\frac{2\,g)\,dx + (f + 2\,h)x^n dx}{2} + \frac{-\,ccdx + 2\,n\,(bc - af)x^n dx - 2\,cfx^n dx - ffx^{2n} dx}{4\,(a + bx^n)} =$$

or $(a + b\,x^n)x^{p+2}$ divisa reducitur ad hanc

$$dy + x^p y^2 dx + \frac{p(p + 2)\,dx}{4\,x^{p+2}} + \frac{(c + 2\,g)\,dx + (f + 2\,h)x^n dx}{2\,(a + bx^n)x^{p+2}}$$
$$= \frac{(c + fx^n)^2 dx - 2\,n\,(bc - af)x^n dx}{4\,(a + bx^n)^2 x^{p+2}}.$$

aequatio ita est comparata, ut posito

$$g = -\,cm - am(m - 1) \ \text{ et } \ h = -\,fk - b\,k(k - 1),$$

sit integrabilis, si fuerit

$$\text{vel} \quad \frac{-\,(k + m - 1)b - f}{bn} \quad \text{vel} \quad \frac{(k + m - 1)a + c}{an}$$

us integer affirmativus.

et aequatio inventa transibit in hanc

$$dy + x^p y^2 dx + \frac{(p+1)^2 dx}{4\,x^{p+2}} - \frac{(a-c)^2 dx}{4\,a^2 x^{p+2}} + \frac{(g+h}{(a+b}$$

quae, si sit

$$g = -cm - am(m-1) \quad \text{et} \quad h = -\frac{b}{a}(ck + a$$

integrabilis existit, si

$$\frac{(k+m-1)a+c}{an}$$

fuerit numerus integer sive affirmativus sive negativus.
quo prodeat ista aequatio

$$dy + x^p y^2 dx + \frac{(p+1)^2 dx}{4\,x^{p+2}} = \frac{(amm + bkk}{(a+bx^n)x}$$

quae integrabilis erit, si $\frac{k+m}{n}$ fuerit numerus integer.

17. Ponamus in aequatione generali ultima § 15 inv
termini simplices prodeant, habebitur ista aequatio

$$dy + x^p y^2 dx + \frac{(p+1)^2 dx}{4\,x^{p+2}} - \frac{(a-c)^2 dx}{4\,aax^{p+2}}$$

$$+ \frac{(af - naf + 2\,ah - cf)x^n dx}{2\,a^2 x^{p+2}} - \frac{ffx^{2n} dx}{4\,aax^{p+2}}$$

quae posito

$$g = -cm - am(m-1) \text{ et } h = -$$

integrabilis existit, si vel

$$\frac{(k+m-1)a+c}{an}$$

fuerit numerus integer affirmativus, vel si sit $f = 0$; qu
constat. Ponamus

$$a^2(p+1)^2 - (a-c)^2 + 4\,ag = \alpha a^2, \quad \text{atque} \quad af - naf$$

erit

$$g = \frac{aa + a(n + 2k + \beta)^2 - a(p+1)^2}{4};$$

s substitutis erit

$$dy + x^p y^2 dx + \frac{a dx}{4 x^{p+2}} + \frac{\beta f x^n dx}{2 a x^{p+2}} - \frac{ff x^{2n} dx}{4 aa x^{p+2}} = 0,$$

ob valorem ipsius g iam ante definitum

$$n + 2k + \beta = 2m + \sqrt{((p+1)^2 - a)},$$

quatio integrationem admittet, si fuerit

$$\frac{m - n - k - \beta}{n} \quad \text{seu} \quad \frac{-n - \beta \pm \sqrt{((p+1)^2 - a)}}{2n}$$

us integer affirmativus. Sit $a = 0$ et $\beta = 0$, habebitur ista aequatio

$$dy + x^p y^2 dx = \frac{ff x^{2n-p-2} dx}{4 aa}$$

toties integrationem admittit, quoties fuerit

$$\frac{-n \pm (p+1)}{2n}$$

us integer affirmativus. Sit ergo

$$i = \frac{-n \pm (p+1)}{2n}, \quad \text{erit} \quad n = \frac{\pm (p+1)}{2i+1};$$

aequatio

$$dy + x^p y^2 dx = \frac{ff x^{\frac{\pm 2(p+1)}{2i+1} - p - 2} dx}{4 aa}$$

r erit integrabilis. Haec autem aequatio ipsa est RICCATIANA[1]; na $p = 0$ prodit

$$dy + y^2 dx = \frac{ff x^{\frac{\pm 2 - 4i - 2}{2i+1}} dx}{4 aa}.$$

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1) Vide notam 1, p. 17. H. D.

quae integrabilis erit, si fuerit

$$\frac{\pm(p+1)-n-\beta}{2n}$$

numerus integer affirmativus puta i. Facto autem

$$\pm(p+1)-n-\beta=2ni \quad \text{erit} \quad \beta=\pm(p+1)-n(2$$

Quamobrem haec aequatio

$$dy+x^p y^2 dx=\frac{ffx^{2n-p-2}dx}{4aa}+\frac{(nf(2i+1)\pm f(p+1))x^{n-}}{2a}$$

semper est integrabilis. Hinc sequuntur sequentes aequationes

$$dy+y^2dx=\frac{ffxxdx}{4aa}+\frac{f(4i+2\pm 1)dx}{2a}$$

$$dy+y^2dx=\frac{ffdx}{4aa}+\frac{f(2i+1\pm 1)dx}{2ax}$$

$$dy+\frac{y^2dx}{x}=\frac{ffxdx}{4aa}+\frac{f(2i+1)dx}{2a}$$

quae omnes sunt integrabiles. Quare haec aequatio

$$dy+Ay^2du=Buudu+Cdu$$

integrabilis existit, quando $\frac{C\sqrt{A}}{\sqrt{B}}$ fuerit numerus integer affirm
namque $4i+2\pm 1$ omnes numeros impares complectitur in se

19. Ponamus in superiore aequatione tantum $\beta=0$; et
aequatio

$$dy+x^p y^2 dx=\frac{ffx^{2n}dx}{4aax^{p+2}}-\frac{adx}{4x^{p+2}},$$

quae integrabilis erit, quoties fuerit

$$\frac{-n\pm\sqrt{((p+1)^2-a)}}{2n}$$

rem haec aequatio

$$dy + x^p y^2 dx = \frac{ffx^{2n-p-2}dx}{4aa} + \frac{(n^2(2i+1)^2 - (p+1)^2)dx}{4x^{p+2}}$$

ntegrabilis erit. Si sit $p = 0$, erit ista aequatio

$$dy + y^2 dx = \frac{ff}{4aa}x^{2n-2}dx + \frac{(n^2(2i+1)^2 - 1)dx}{4xx}$$

emper integrabilis. Hinc ponendo $\frac{ff}{4aa} = A$, quia f et a sunt quanti-
bitrariae, integrabiles erunt sequentes aequationes

$$dy + y^2 dx = A\,dx + \frac{i(i+1)dx}{xx}$$

$$dy + y^2 dx = Ax^2 dx + \frac{(4i+3)(4i+1)dx}{4xx}$$

$$dy + y^2 dx = Ax^4 dx + \frac{(3i+2)(3i+1)dx}{xx}$$

ius generis innumerabiles aliae.

Fiat in aequatione

$$dy + x^p y^2 dx = \frac{dx(ffx^{2n} - 2a\beta fx^n - \alpha a^2)}{4a^2x^{p+2}}$$

nta $\alpha = -\beta^2$, quo sit

$$dy + x^p y^2 dx = \frac{(fx^n - \beta a)^2 dx}{4a^2 x^{p+2}},$$

uatio toties integrabilis erit, quoties fuerit

$$\frac{-n - \beta \pm \sqrt{((p+1)^2 + \beta^2)}}{2n}$$

integer affirmativus puta $= i$. Erit ergo

$$(2i+1)n + \beta = \sqrt{((p+1)^2 + \beta^2)}$$

$$\beta = \frac{(p+1)^2 - n^2(2i+1)^2}{2n(2i+1)};$$

quoties ergo β huiusmodi habuerit valorem, aequatio

$$dy + x^p y^2 dx = \frac{(fx^n - \beta a)^2 dx}{4a^2 x^{p+2}}$$

integrationem admittet. Posito igitur $p = 0$ ista aequatio

$$dy + y^2 dx = \frac{dx}{xx}\left(\frac{n^2(2i+1)^2 - 1}{4n(2i+1)} + \frac{f}{2a}x^n\right)^2$$

integrabilis erit. At si $p = -1$ prodibit ista aequatio

$$dy + \frac{y^2 dx}{x} = \frac{dx}{x}\left(\frac{n(2i+1)}{4} + \frac{f}{2a}x^n\right)^2$$

integrabilis. Sit autem $x^{p+1} = t$, erit

$$x^p dx = \frac{dt}{p+1}, \quad x^n = t^{\frac{n}{p+1}} \quad \text{et} \quad \frac{dx}{x^{p+2}} = \frac{dt}{(p+1)tt},$$

habebitur ergo ista aequatio

$$(p+1)dy + y^2 dt = \frac{(ft^{\frac{n}{p+1}} - \beta a)^2 dt}{4a^2 tt}$$

quae integrabilis erit, si fuerit

$$\beta = \frac{(p+1)^2 - n^2(2i+1)^2}{2n(2i+1)}.$$

21. Multo quidem plura consectaria ex nostra aequatione g
parum elegantia deduci possent, sed ampliorem evolutionem aliis
iuvant, relinquo. Interim notari convenit praeter hanc methodum,
secutus, alias dari innumeras, quarum ope aequationes different
certis duntaxat casibus integrabiles evadunt, inveniri possunt, s
nimis fit laboriosa. Ita si consideretur[1]) haec aequatio

---

1) Vide L. EULERI Commentationes 274 et 710: *Constructio aequationis different*

$$Aydu^2 + (B + Cu)du\,dy + (D + Eu + Fuu)ddy = 0$$

sumto elemento du constante. Novi comment. acad. sc. Petrop. 8 (1760/1), 1763, p. 1
transformationis singularis serierum. Nova acta acad. sc. Petrop. 12 (1794), 1801, p.
EULERI *Opera omnia*, I 22 et I 16.

$$v = Ax^m + Bx^{m+n} + Cx^{m+2n} + \text{etc.},$$

fficientes quidem definire licebit, sed binos contiguos evanescere oporte quentes omnes evanescant. Scilicet quo fiat $v = Ax^m$ necesse est

$$p + fm + am(m-1) = 0$$

ac

$$q + gm + bm(m-1) = 0 \quad \text{et} \quad r + hm + cm(m-1) = 0.$$

utem fiat

$$v = Ax^m + Bx^{m+n},$$

ur ut sit

$$B = -\frac{A(q + gm + bm(m-1))}{nf + na(2m + n - 1)},$$

)

$$p + fm + am(m-1) = 0,$$

$$r + h(m+n) + c(m+n)(m+n-1) = 0$$

cto $\quad n^2(h + c(2m+n-1))(f + a(2m+n-1))$
$+ gm + bm(m-1))(q + g(m+n) + b(m+n)(m+n-1)) = 0.$

satis liquet, ulterius progrediendo laborem in immensum excrescere.

Unicum tamen coronidis loco exemplum simplicius afferam, quo fo

$$b = 0, \ c = 0, \ f = 0 \ \text{ et } \ g = 0,$$

$$\text{positoque} \quad v = e^{\int z dx} \quad \text{posui} \quad z = y - \frac{h}{2a}x^{2n-1},$$

cto sequens provenit aequatio

$$^2dx = \frac{hh}{4aa}x^{4n-2}dx + \frac{x^{2n-2}dx}{2a}(h(2n-1) - 2r) - \frac{q}{a}x^{n-2}dx + \frac{m(m-1)d}{xx}$$

er duos casus expositos integrabilis est,

secundo si fuerit $q = h\sqrt{ah(r - h - 2hn)}$ et $r = - h(hn + n)$;

practer hos vero casus infiniti dantur alii, quibus ista aequatio pariter i
bilis existit, sed ad eos determinandos resolutiones aequationum p
dimensionum requiruntur. Posito

$$r = \frac{h(2n - 1)}{2}$$

per secundum casum ista aequatio

$$dy + y^2 dx = \frac{hh}{4aa} x^{4n-2} dx \pm \frac{n}{a} x^{n-2} dx \sqrt{3ahn} + \frac{(16nn - 1) dx}{4xx}$$

integrabilis erit.

METHODUS AEQUATIONES DIFFERENTIALE ALTIORUM GRADUUM INTEGRANDI ULTERIU PROMOTA

Commentatio 188 indicis ENESTROEMIANI

Novi Commentarii academiae scientiarum Petropolitanae 3 (1750/1), 1753, p. 3–35

Summarium ibidem p. 6–8

SUMMARIUM

Haec Dissertatio sine dubio insigne continet calculi integralis augmentum; c radatur methodus, innumerabiles aequationes altiorum graduum ita expedite di, ut per unam operationem statim aequatio integralis obtineatur, neque op integrationes successive instituere, quot est gradus aequatio differentialis prop smodi operationes aliae methodi adhuc cognitae requirant. Tradiderat autem A olumine Septimo Miscellaneorum Berolinensium iam specimen huius method erat una operatione integrale huius aequationis invenire:

$$0 = Ay + \frac{Bdy}{dx} + \frac{Cddy}{dx^2} + \frac{Dd^3y}{dx^3} + \frac{Ed^4y}{dx^4} + \frac{Fd^5y}{dx^5} + \text{etc.},$$

elementum dx sumtum est constans, litterae autem A, B, C, D etc. coefficientes quoscunque constantes; nunc autem hanc methodum extendit ad hanc fo te latius patentem:

$$X = Ay + \frac{Bdy}{dx} + \frac{Cddy}{dx^2} + \frac{Dd^3y}{dx^3} + \frac{Ed^4y}{dx^4} + \frac{Fd^5y}{dx^5} + \text{etc.},$$

littera X denotat quantitatem quamcunque ex variabili x et constantibus ute latam. Omnino hic notatu est dignum, quod operatio semper succedat, ad c que etiam gradum differentialium aequatio ascendat, ne gradu quidem infinit uso, cuius eximia exempla Auctor in sequentibus exhibet. In hac autem Dissert num casum admodum simplicem hac aequatione $d^3y = ydx^3$ contentum me ari persequitur, ostendens quam prolixum ac taediosum calculum eius solutio req pe quo tandem ad aequationem quidem differentialem primi ordinis perdu

... integratio gravibus adhuc premitur difficultatibus. Inc
in subsidium vocatis artificiis elicit integrale quidem, sed tantum
denique per novam operationem integrale completum colligit. Tum
integrationes instituere oportet, antequam solutio ad finem sit perdr
iudicium de praestantia novae methodi ferre licebit, cuius beneficio
molestis ambagibus una eaque facillima operatione non solum hae
sed generalis exhibita ita perfecte resolvitur, ut statim aequatio
reperiatur. Operatio autem illa reducitur ad resolutionem aequation
forma ita ex proposita aequatione differentiali derivatur, ut sit

$$0 = A + Bz + Cz^2 + Dz^3 + Ez^4 + Fz^5 + \text{etc.}$$

atque nunc totum negotium in resolutione huius aequationis Alge
quod quidem cum de integratione est quaestio merito pro facillimo hab
aequationis cunctae quaerendae sunt radices, earumque quaelibet sup
simplicissimae portionem integralis quaesiti, ita ut omnibus radicibus b
universum integrale completum obtineatur. Difficultate quidem haec
videtur iis casibus, quibus illa aequatio Algebraica radices habet vel a
biles; sed et huic incommodo feliciter occurrit Auctor, dum pro his
praebet regulas, quarum ope tota operatio aeque expedite perfici p

Si quis quaerat, quemnam usum huiusmodi speculationes, quae
nimis steriles videantur, habere queant, ei audacter respondere licet,
Problema Physicum, vel ad vitam communem pertinens, cuius solu
plerumque ad aequationem differentialem altioris cuiusdam ordinis p
facile intelligere licet, quam parum tales speculationes contemni me

1. Tradidi in volumine septimo Miscellaneorum Berolinensi
facilem aequationes differentiales cuiusque gradus, in quibus
ubique unicam obtinet dimensionem, alterius vero tantum dif
constans assumitur, occurrit, integrandi, atque adeo aequa
quae differentialem propositam penitus exhauriat, inveniend
si aequatio proposita differentialis primum gradum superet, p
integrationibus opus erat, sed uno quasi ictu cuiuscunque demu
aequatio proposita, methodus ibi exposita eandem suppedita
finitam, quae proditura esset, si successive tot instituerentu
quot gradus differentialia in ea obtinent. Sic si aequatio pro
rentialis quarti gradus, more solito ea per unam integrationem p
tionem differentialem tertii gradus reduci, tum vero denuo in
deberet, ut ad gradum secundum revocetur: quo facto adhuc d

1) Commentatio 62, p. 108 huius voluminis.

tiplicitatem per methodum meam prorsus evito, dum unica opera
im veram aequationem integralem elicio.

2. Quantopere autem modum integrandi vulgarem toties repeten
ties differentialitas in aequatione inest, secuti in molestissimos cal
damus, unico exemplo ostendisse iuvabit[1]). Sit ergo proposita haec aeq
erentialis tertii gradus

$$d^3y = ydx^3,$$

qua elementum dx constans ponitur. Haec aequatio, etsi mea met
llime ter integratur, tamen ne quidem modus eam semel tantum integ
spicitur. Statim quidem, quia variabilis x ipsa deest, apparet ea
dum secundum deprimi posse. Si enim ponatur $dx = pdy$, ob dx con

$$0 = pddy + dpdy$$

denuo differentiando

$$0 = pd^3y + 2dpddy + dyddp.$$

de fit

$$ddy = -\frac{dpdy}{p}$$

$$d^3y = -\frac{2dpddy}{p} - \frac{dyddp}{p} = \frac{2dp^2dy}{pp} - \frac{dyddp}{p},$$

valores in aequatione proposita $d^3y = ydx^3$ substituti dabunt:

$$\frac{2dp^2dy}{pp} - \frac{dyddp}{p} = yp^3dy^3 \quad \text{seu} \quad yp^4dy^2 = 2dp^2 - pddp.$$

ne cum neque dp neque dy sit constans, sed constantiae ratio ex aequa
$= -\frac{dpdy}{p}$ definiatur, per methodos solitas vix ulterius tractari p
nsmutari quidem aequatio potest in aliam formam, in qua nullum
tiale constans insit. Ponatur $dp = qdy$; erit

$$ddp = qddy + dqdy$$

---

1) Cf. Commentationem 62, § 1, p. 108.

unde
$$ddp = -\frac{qqdy^2}{p} + dqdy$$

sicque aequatio inventa hanc induet formam:

$$yp^5dy = 2qqdy + qqdy - pdq = 3qqdy - pdq$$

In qua pro lubitu differentiale constans assumere licet. Sit dy ... $q = \frac{dp}{dy}$ erit $dq = \frac{ddp}{dy}$; habebiturque

$$yp^5dy^2 = 3dp^2 - pddp.$$

At si ponatur $p = \frac{1}{r}$ fiet

$$ydy^2 = rdr^2 + rrddr,$$

quae aequatio cum ambae variabiles ubique totidem scilicet tr... teneant, ope methodi meae[1]) in III. Tomo Commentariorum exp... potest. Ponatur scilicet

$$y = e^{\int zdu} \text{ et } r = e^{\int zdu}u$$

denotante e numerum cuius logarithmus hyperbolicus $= 1$, e...

$$dy = e^{\int zdu}zdu \text{ et } ddy = 0 = e^{\int zdu}(zddu + dudz + z \ldots$$

Deinde est

$$dr = e^{\int zdu}(du + zudu)$$

et ob $r = uy$ erit

$$ddr = 2dudy + yddu = e^{\int zdu}(ddu + 2zdu^2).$$

Sed $ddu = -\frac{dudz}{z} - zdu^2$, unde

$$ddr = e^{\int zdu}\left(zdu^2 - \frac{dudz}{z}\right).$$

Qui valores in aequatione

$$ydy^2 = rdr^2 + rrddr$$

substituti dabunt:

$$zzdu = u(1 + zu)^2du + uuzdu - \frac{uudz}{z},$$

---

1) Vide Commentationem 10 § 11; p. 6 huius voluminis.

$$\frac{dt}{t} = ttu^3du + 3tudu - ttdu.$$

otius cum aequatio proposita ipsa facile conficiatur, inde integrati
equationis petenda videtur. Ponatur porro $t = \frac{1}{s}$, atque aequatio in
bibit in hanc

$$sds + 3sudu = du(1 - u^3),$$

equatio immediate ex proposita elicitur, ponendo

$$dx = \frac{du}{s} \quad \text{et} \quad \frac{dy}{y} = \frac{udu}{s},$$

m ob $\frac{du}{s}$ constans,

$$sddu = dsdu \quad \text{et} \quad \frac{ddy}{y} = \frac{u^2du^2}{ss} + \frac{du^2}{s}$$

$$\frac{d^3y}{y} = \frac{u^3du^3}{s^3} + \frac{3udu^3}{ss} + \frac{duddu}{s} = \frac{u^3du^3}{s^3} + \frac{3udu^3}{ss} + \frac{du^2ds}{ss},$$

ores in aequatione $d^3y = ydx^3$ substituti praebebunt aequationem in

$$sds + 3sudu = du(1 - u^3).$$

Totum ergo negotium ad integrationem huius aequationis revocatur
ntegrabilem esse vel inde patet, quod aequatio differentialis tertii gra
qua est nata, integrationem admittat. Quemadmodum autem hoc opu
olvendum, [ostendatur] in aequatione latius patente, quae per eanden
ationem ex hac aequatione differentiali tertii gradus oritur,

$$Aydx^3 + Bdx^2dy + Cdxddy + Dd^3y = 0.$$

it autem ponendo

$$dx = \frac{du}{s} \quad \text{et} \quad \frac{dy}{y} = \frac{udu}{s}$$

equatio differentialis primi gradus

$$Dsds + sdu(C + 3Du) + du(A + Bu + Cuu + Du^3) = 0,$$

quam primum obser...

Erit enim $ds = \beta du + 2\gamma u du$. Unde fit

$$\begin{aligned}\frac{Dsds}{du} &= D\alpha\beta + 2D\alpha\gamma u + 2D\beta\gamma u \ldots \\ &\quad + D\beta\beta u \quad + D\beta\gamma u^2 \\ s(C + 3Du) &= C\alpha + C\beta u \quad + C\gamma u u \\ &\quad + 3D\alpha u \quad + 3D\beta u^2 \\ A + Bu + Cu^2 + Du^3 &= A + Bu \quad + Cu^2\end{aligned}$$

Reddantur iam singuli termini homologi $= 0$, fietque prim...
Unde fit vel $1 + \gamma = 0$ vel $1 + 2\gamma = 0$. Deinde est

$$3D\beta(\gamma + 1) + C(\gamma + 1) = 0,$$

cui aequationi quoque satisfacit $\gamma + 1 = 0$, ergo erit $\gamma = \ldots$

$$D\alpha = -B - C\beta - D\beta\beta \quad \text{seu} \quad \alpha = \frac{-B - C\ldots}{\ldots}$$

Substituatur hic valor in aequatione

$$D\alpha\beta + C\alpha + A = 0 \quad \text{seu} \quad D^2\alpha\beta + CD\alpha + \ldots$$

eritque

$$\begin{aligned}&-BD\beta - CD\beta^2 - DD\beta^3 = 0\\ -&BC - CC\beta - CD\beta^2\\ +&AD\end{aligned}$$

Ad β ergo inveniendum hanc aequationem cubicam res...
autem α quaeratur, erit:

$$D^2\alpha^3 + BD\alpha^2 + AC\alpha + A^2 = 0.$$

Sit $\alpha = \frac{A\omega}{D}$, fiet $A\omega^3 + B\omega^2 + C\omega + D = 0$

Sit ergo ω radix huius aequationis cubicae, fiet

$$\alpha = \frac{A\omega}{D}, \quad \beta = \frac{-D - C\omega}{D\omega} \quad \text{et} \quad \gamma = -\ldots$$

atque

$$s = \frac{A\omega^2 - (D + C\omega)u - D\omega u^2}{D\omega}.$$

Porro fiet

$$x = \int \frac{du}{s} = \int \frac{D\omega du}{A\omega^2 - (D + C\omega)u - D\omega u\ldots}$$

atque

$$ly = \int \frac{udu}{s} = \int \frac{D\omega u du}{A\omega^2 - (D + C\omega)u - D\omega\ldots}$$

n eo nulla nova occurrit constans, quae in ipsa aequatione non insi
cognito valore particulari ipsius s, ex eo valor completus sequenti mod
. Ponatur valor iam inventus

$$\frac{A\omega^2 - (D + C\omega)u - D\omega u^2}{D\omega} = V$$

atur $s = V + z$, ut sit

$$ds = dV + dz,$$

prodibit

$$\left.\begin{array}{llll} DVdV & + DVdz + DzdV & + Dzdz & \\ + CVdu & & + Czdu & \\ + 3\,DVudu & & + 3\,Duzdu & \\ + (A + Bu + Cuu + Du^3)du & & & \end{array}\right\} = 0.$$

ore sit per hypothesin

$$DVdV + Vdu(C + 3\,Du) + du(A + Bu + Cu^2 + Du^3) = 0,$$

$$Dzdz + z(Cdu + 3\,Dudu + DdV) + DVdz = 0.$$

$$V = \frac{A\omega}{D} - \frac{u}{\omega} - \frac{Cu}{D} - uu$$

$$dV = -\frac{du}{\omega} - \frac{Cdu}{D} - 2\,udu$$

$$Dzdz + z\left(-\frac{Ddu}{\omega} + Dudu\right) + \frac{dz}{\omega}(A\omega^2 - (D + C\omega)u - D\omega u^2) = 0$$

$$zdz + zdu\left(u - \frac{1}{\omega}\right) + dz\left(\frac{A\omega}{D} - \frac{(D + C\omega)u}{D\omega} - uu\right) = 0,$$

equatio nisi bene tractetur, difficulter ad separationem variabilium pe
. Interim tamen continetur in hac forma generali, quae separatione
it:

$$zdz + zdu(u + a) = dz(uu + 2\,bu + c).$$

et differentiando:

$$dz - pdu = \frac{(u+a)(uu+2bu+c)dp + pdu(2p(u+b) + uu + 2}{(p+u+a)^2}$$

seu

$$pdu(pp + 2ap - 2bp + aa - 2ab + c) = (u + a)(uu + 2$$

in qua variabiles sponte a se invicem separantur; erit enim:

$$\frac{dp}{p(pp + 2(a-b)p + aa - 2ab + c)} = \frac{du}{(u+a)(uu + 2bu +}$$

Opus autem foret summe taediosum, si hanc aequationem in exinde integrale aequationis differentialis tertii gradus eruere v

4. Apparet hinc quanto labore tandem huiusmodi reg integrale aequationis differentialis tertii gradus erui possit, methodi meae in Volumine septimo Miscellaneorum expositae n perspicitur. Eo magis autem eius utilitas in oculos incurret, si lo differentialis tertii gradus alia, quae sit quarti altiorisve gradus tractetur, tum enim substitutiones hic adhibitae aequationem non primi, sed secundi altiorisve gradus praebebit, cuius int artificiis obtineri poterit. Et quamvis tandem etiam huius aequa inveniretur, tamen id plerumque tantum foret particulare, et p mas demum substitutiones suppeditat, et ipsius aequationis p grale, et quidem particulare tantum: cum mea methodus fere s statim integrale completum praebeat. Quod ut clarius intelligatu tradita substitutione in hac aequatione differentiali quarti gra

$$Aydx^4 + Bdx^3dy + Cdx^2ddy + Ddxd^3y + Ed^4y =$$

in qua dx ponitur constans. Sit igitur

$$dx = \frac{du}{s} \quad \text{seu} \quad du = sdx, \quad \text{et} \quad \frac{dy}{y} = \frac{udu}{s} = udx;$$

erit ob dx constans:

$$\frac{ddy}{y} - \frac{dy^2}{y^2} = dxdu = sdx^2;$$

Hinc fiet porro

$$\frac{d^3y}{y} - \frac{dy\,ddy}{y^2} = 2\,usdx^3 + dsdx^2 \quad \text{et} \quad \frac{d^3y}{y} = u^3dx^3 + 3\,usdx^3 + d\ldots$$

iterumque differentiando prodibit

$$\frac{d^4y}{y} - \frac{dyd^3y}{yy} = 3\,uusdx^4 + 3\,udx^3ds + 3\,ssdx^4 + dx^3dds\,,$$

ideoque

$$\frac{d^4y}{y} = u^4dx^4 + 6\,uusdx^4 + 4\,udx^3ds + 3\,ssdx^4 + dx^2dds.$$

Quibus valoribus in aequatione hac substitutis

$$Adx^2 + \frac{Bdxdy}{y} + \frac{Cddy}{y} + \frac{Dd^3y}{ydx} + \frac{Ed^4y}{ydx^2} = 0$$

proveniet haec aequatio:

$$\begin{aligned} Adx^2 + Budx^2 + Cu^2dx^2 + Csdx^2 + Du^3dx^2 + 3\,Dusdx^2 + Dd\ldots \\ + Eu^4dx^2 + 6\,Euusdx^2 + 4\,Eudxds + 3\,Essdx^2 + Edds = 0\ldots \end{aligned}$$

Cum autem sit $dx = \frac{du}{s}$, erit

$$\begin{aligned} du^2(A + Bu + Cu^2 + Du^3 + Eu^4) + sdu^2(C + 3\,Du + 6\,Euu) + 3\ldots \\ + sduds(D + 4\,Eu) + Essdds = 0\,. \end{aligned}$$

Apparet quidem huic aequationi satisfieri, si sit $s = 0$ et u radix huiu[s aequa-]
tionis:

$$A + Bu + Cu^2 + Du^3 + Eu^4 = 0.$$

Sit ergo α una ex radicibus huius aequationis, et sumendo $u = \ldots$ $\frac{dy}{y} = \alpha dx$ et $y = e^{\alpha x}$, qui valor quoque aequationi differentiali quart[ae] propositae conveniet. Erit autem tantum integrale maxime particulare[.] autem quaternae aequationis $A + Bu + Cu^2 + Du^3 + Eu^4 = 0$ radic[es] sint $\alpha, \beta, \gamma, \delta$, suppeditare queant valorem

$$y = \mathfrak{A}e^{\alpha x} + \mathfrak{B}e^{\beta x} + \mathfrak{C}e^{\gamma x} + \mathfrak{D}e^{\delta x},$$

qui est integrale completum, tamen hinc non facile patet, qualis fu[turus sit] valor ipsius y, si radicum $\alpha, \beta, \gamma, \delta$ quaedam fuerint imaginariae vel

aequationis differentio-differentialis inter u et s assignabitur. F

$$u = \frac{dy}{ydx} \quad \text{et} \quad s = \frac{du}{dx};$$

ideoque

$$u = \frac{\mathfrak{A}\alpha e^{\alpha x} + \mathfrak{B}\beta e^{\beta x} + \mathfrak{C}\gamma e^{\gamma x} + \mathfrak{D}\delta e^{\delta x}}{\mathfrak{A}e^{\alpha x} + \mathfrak{B}e^{\beta x} + \mathfrak{C}e^{\gamma x} + \mathfrak{D}e^{\delta x}}$$

et

$$s = \frac{\mathfrak{A}\mathfrak{B}(\alpha-\beta)^2 e^{(\alpha+\beta)x} + \mathfrak{A}\mathfrak{C}(\alpha-\gamma)^2 e^{(\alpha+\gamma)x} + \mathfrak{A}\mathfrak{D}(\alpha-\delta)^2 e^{(\alpha+\delta)x} + \mathfrak{B}\mathfrak{C}(\beta-\gamma}{(\mathfrak{A}e^{\alpha x} + \mathfrak{B}e^{\beta x} + \mathfrak{C}e^{\gamma x} + \mathfrak{D}e^{\delta x})^2}$$

Hinc concluditur fore:

$$s + uu = \left\{ \begin{array}{l} \dfrac{\mathfrak{A}^2\alpha^2 e^{2\alpha x} + \mathfrak{B}^2\beta^2 e^{2\beta x} + \mathfrak{C}^2\gamma^2 e^{2\gamma x} + \mathfrak{D}^2\delta^2 e^{2\delta x}}{(\mathfrak{A}e^{\alpha x} + \mathfrak{B}e^{\beta x} + \mathfrak{C}e^{\gamma x} + \mathfrak{D}e^{\delta x})^2} + \\ \dfrac{\mathfrak{A}\mathfrak{B}(\alpha^2+\beta^2)e^{(\alpha+\beta)x} + \mathfrak{A}\mathfrak{C}(\alpha^2+\gamma^2)e^{(\alpha+\gamma)x} + \text{et}}{(\mathfrak{A}e^{\alpha x} + \mathfrak{B}e^{\beta x} + \mathfrak{C}e^{\gamma x} + \mathfrak{D}e^{\delta x})^2} \end{array} \right.$$

quae fractio deprimi potest, eritque

$$s + uu = \frac{\mathfrak{A}\alpha^2 e^{\alpha x} + \mathfrak{B}\beta^2 e^{\beta x} + \mathfrak{C}\gamma^2 e^{\gamma x} + \mathfrak{D}\delta^2 e^{\delta x}}{\mathfrak{A}e^{\alpha x} + \mathfrak{B}e^{\beta x} + \mathfrak{C}e^{\gamma x} + \mathfrak{D}e^{\delta x}}.$$

Cum iam sit

$$u = \frac{\mathfrak{A}\alpha e^{\alpha x} + \mathfrak{B}\beta e^{\beta x} + \mathfrak{C}\gamma e^{\gamma x} + \mathfrak{D}\delta e^{\delta x}}{\mathfrak{A}e^{\alpha x} + \mathfrak{B}e^{\beta x} + \mathfrak{C}e^{\gamma x} + \mathfrak{D}e^{\delta x}},$$

si hinc x, quod autem actu fieri nequit, eliminetur, prodibit aequa
Si quidem ponatur $\mathfrak{C} = 0$ et $\mathfrak{D} = 0$, prodibit aequatio integral
haec

$$s + uu - (\alpha + \beta) u + \alpha\beta = 0.$$

Quare si fuerint α et β duae radices huius aequationis

$$A + Bu + Cu^2 + Du^3 + Eu^4 = 0,$$

aequationi differentio-differentiali inter s et u satisfaciet hic v

$$s = -\alpha\beta + (\alpha + \beta) u - uu.$$

In aequatione autem illa non du sed $\frac{du}{s}$ positum est constans, qu

exuetur ponendo $ds = qdu$; erit enim $\frac{ds}{qs}$ constans ideoque

$$qsdds = qds^2 + sdsdq, \text{ et } dds = \frac{ds^2}{s} + \frac{ds\,dq}{q};$$

$$dq = \frac{\quad}{du} \quad \frac{\quad}{q} = \frac{\quad}{ds},$$

e fit[1])

$$dds = \frac{ds^2}{s} + dds.$$

dibit ergo haec aequatio:

$$^2(A + Bu + Cu^2 + Du^3 + Eu^4) + sdu^2(C + 3Du + 6Eu^2) + 3Es\ldots$$
$$+ sduds(D + 4Eu) + Esds^2 + Essdds = 0$$

qua differentiale du assumtum est constans. Quodsi iam formulae

$$A + Bu + Cu^2 + Du^3 + Eu^4$$

or trinomialis sit

$$L + Mu + Nu^2$$

integrale particulare

$$L + Mu + Nu^2 + Ns = 0.$$

5. Quoniam autem hic methodum meam integrandi aequationes diff…es altiorum graduum ulterius extendere constitui, regulam quam loco c…i paucis repetam. Patet vero methodus mea ad omnes aequationes in…na generali contentas:

$$0 = Ay + \frac{Bdy}{dx} + \frac{Cddy}{dx^2} + \frac{Dd^3y}{dx^3} + \frac{Ed^4y}{dx^4} + \frac{Fd^5y}{dx^5} + \text{etc.},$$

differentiale dx positum est constans. Ad huius aequationis inte…is terminis expressum inveniendum ex ea formetur sequens forma …ica:

$$A + Bz + Cz^2 + Dz^3 + Ez^4 + Fz^5 + Gz^6 + \text{etc.},$$

…us quaerantur omnes factores reales tam simplices quam trinomiales, …s, si qui fuerint inter se aequales, coniunctim repraesententur. Ex que…em factore nascetur integralis pars, et, si omnes istae partes ex sin…toribus oriundae in unam summam coniiciantur, habebitur integrale

1) In hac formula dds significationes dissimiles habet. In priore membro $\frac{du}{s}$ positum est co… …steriore du positum est constans.

Factores	Partes Integralis
$z - k$	αe^{kx}
$(z - k)^2$	$(\alpha + \beta x) e^{kx}$
$(z - k)^3$	$(\alpha + \beta x + \gamma x^2) e^{kx}$
$(z - k)^4$	$(\alpha + \beta x + \gamma x^2 + \delta x^3) e^{kx}$
etc.	etc.
$zz - 2kz \cos. \Phi + kk$	$\alpha e^{kx \cos. \Phi} \sin. kx \sin. \Phi + \mathfrak{A} e^{kx \cos. \Phi}$ co
$(zz - 2kz \cos. \Phi + kk)^2$	$(\alpha + \beta x) e^{kx \cos. \Phi} \sin. kx \sin. \Phi$ $+ (\mathfrak{A} + \mathfrak{B} x) e^{kx \cos. \Phi} \cos. kx \sin. \Phi$
$(zz - 2kz \cos. \Phi + kk)^3$	$(\alpha + \beta x + \gamma x^2) e^{kx \cos. \Phi} \sin. kx \sin. \Phi$ $+ (\mathfrak{A} + \mathfrak{B} x + \mathfrak{C} x^2) e^{kx \cos. \Phi} \cos. kx$ s
$(zz - 2kz \cos. \Phi + kk)^4$	$(\alpha + \beta x + \gamma x^2 + \delta x^3) e^{kx \cos. \Phi} \sin.$ kx $+ (\mathfrak{A} + \mathfrak{B} x + \mathfrak{C} x^2 + \mathfrak{D} x^3) e^{kx \cos. \Phi}$ c
etc.	etc.

In his formulis litterae $\alpha, \beta, \gamma, \delta$ etc., $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{D}$ etc. denot
quantitates arbitrarias. Hinc in partibus integralis colligendis ca
eadem harum litterarum bis scribatur, quia alioquin extensio in
geretur. Oportebit ergo has constantes continuo novis litteris i
modo in aequationem integralem tot ingredientur constantes ar
gradus fuerit aequatio differentialis proposita: id quod certu
integrale hoc modo inventum esse completum, atque in aequati
nihil contineri, quod non simul in hac aequatione integrali co
rum in eo loco[1]), ubi hanc methodum fusius exposui, pluribu
illustravi, ita ut circa eius applicationem nulla difficultas locum

6. Aequatio autem generalior, cuius integrationem hic s
denotante X functionem quamcunque ipsius x ita se habet[2]):

$$X = Ay + \frac{Bdy}{dx} + \frac{Cddy}{dx^2} + \frac{Dd^3y}{dx^3} + \frac{Ed^4y}{dx^4} + \text{etc.},$$

in qua iterum differentiale dx constans est assumtum. Hanc igit
quotcunque constet terminis, seu ad quemcunque ea different

1) Vide p. 111 huius voluminis.

2) Vide praeter notam 2 p. 3 huius voluminis adiectam etiam *Institutio*
vol. II, § 856—860, 865—868, 1138—1165, 1172—1224; LEONHARDI EULERI *Opera* c

it functio rationalis integra ipsius x, seu si habeat huiusmodi formam:

$$X = \alpha + \beta x + \gamma x^2 + \delta x^3 + \text{etc.}$$

enim functio X ita sit comparata, adhibeatur huiusmodi substitutio:

$$\begin{aligned}
y &= \mathfrak{A} + \mathfrak{B}x + \mathfrak{C}x^2 + \mathfrak{D}x^3 + \text{etc.} + v\\
\frac{dy}{dx} &= \mathfrak{B} + 2\,\mathfrak{C}x + 3\,\mathfrak{D}x^2 + \text{etc.} + \frac{dv}{dx}\\
\frac{ddy}{dx^2} &= 2\,\mathfrak{C} + 6\,\mathfrak{D}x + \text{etc.} + \frac{ddv}{dx^2}\\
\frac{d^3y}{dx^3} &= 6\,\mathfrak{D} + \text{etc.} + \frac{d^3v}{dx^3}\\
\frac{d^4y}{dx^4} &= \text{etc.} \qquad + \frac{d^4v}{dx^4}\\
&\text{etc.}
\end{aligned}$$

s autem esse $X = \alpha + \beta x + \gamma x^2 + \delta x^3$, atque in valore ipsius y omnes post $\mathfrak{D}x^3$ evanescentes erunt ponendi. Facta ergo substitutione habe-

$$\begin{aligned}
&x + \gamma x^2 + \delta x^3 = \\
&+ \mathfrak{B}Ax + \mathfrak{C}Ax^2 + \mathfrak{D}Ax^3 + Av + \frac{Bdv}{dx} + \frac{Cddv}{dx^2} + \frac{Dd^3v}{dx^3} + \frac{Ed^4v}{dx^4} + \text{etc.}\\
&+ 2\,\mathfrak{C}Bx + 3\,\mathfrak{D}Bx^2\\
&+ 6\,\mathfrak{D}Cx
\end{aligned}$$

coefficientes $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, $\mathfrak{D}$ ita definiri poterunt, ut omnes termini, in on inest v eiusve differentialia, evanescant, fiet enim:

$$\begin{aligned}
&- \frac{3\,\mathfrak{D}B}{A} = \frac{\gamma}{A} - \frac{3\,\delta B}{AA}\\
&- \frac{2\,\mathfrak{C}B}{A} - \frac{6\,\mathfrak{D}C}{A} = \frac{\beta}{A} - \frac{2\gamma B}{A^2} + \frac{6\,\delta B^2}{A^3} - \frac{6\,\delta C}{AA}\\
&- \frac{\mathfrak{B}B}{A} - \frac{2\,\mathfrak{C}C}{A} - \frac{6\,\mathfrak{D}D}{A} = \frac{\alpha}{A} - \frac{\beta B}{A^2} + \frac{2\gamma B^2}{A^3} + \frac{12\,\delta BC}{A^3} - \frac{6\,\delta B^3}{A^4} - \frac{2\gamma C}{A^2} - \frac{6\,\delta D}{A^2}\text{ 1)}.
\end{aligned}$$

1) editione principe $\frac{1\,\delta BD}{A^3}$ loco $\frac{12\,\delta BC}{A^3}$. Correxit H. D.

$$0 = Av + \frac{\ldots}{dx} + \frac{\ldots}{dx^2} + \frac{\ldots}{dx^3} + \frac{\ldots}{dx^4} + \text{etc.},$$

quae aequatio ope superioris methodi integrabitur.

7. Quo autem facilius aequationis propositae, qualiscumq
functio ipsius x, integrale eruamus, a casibus simplicioribus in
primo quidem sit aequatio tantum differentialis primi gradus,

$$X = Ay + \frac{Bdy}{dx},$$

quam patet integrabilem reddi posse, si multiplicetur per huiusm
$e^{\alpha x}dx$ denotante e numerum cuius logarithmus hyperbolicus $= 1$.

$$e^{\alpha x}Xdx = Ae^{\alpha x}ydx + Be^{\alpha x}dy.$$

Atque α ita comparatum esse oportet, ut pars posterior sit differe
piam quantitatis finitae: quae ex termino ultimo alia esse nequi
cuius differentiale cum sit $= Be^{\alpha x}dy + \alpha Be^{\alpha x}ydx$, necesse est ut s
$\alpha = \frac{A}{B}$. Hoc ergo valore pro α sumto erit

$$\int e^{\alpha x}Xdx = Be^{\alpha x}y \quad \text{et} \quad y = \frac{\alpha}{A}e^{-\alpha x}\int e^{\alpha x}Xdx.$$

8. Sit aequatio proposita differentialis secundi gradus[1]):

$$X = Ay + \frac{Bdy}{dx} + \frac{Cddy}{dx^2}.$$

Multiplicetur ea per $e^{\alpha x}dx$ ac definiatur α ita, ut integratio succ
bitur ergo

$$e^{\alpha x}Xdx = Ae^{\alpha x}ydx + Be^{\alpha x}dy + \frac{Ce^{\alpha x}ddy}{dx},$$

cuius integrale sit:

$$\int e^{\alpha x}Xdx = e^{\alpha x}\left(A'y + \frac{B'dy}{dx}\right).$$

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1) Cf. *Institutiones calculi integralis* vol. II, § 856—860, 865—868, 1143—114
p. 192 huius voluminis.

$$e^{\alpha x}Xdx = e^{\alpha x}\left(\alpha A'ydx + A'dy + \frac{B'ddy}{dx} + \alpha B'dy\right).$$

omparatione ergo facta fiet

$$B' = C,\ A' = B - \alpha C \text{ et } A = \alpha B - \alpha^2 C,$$

ebet ergo esse α radix huius aequationis

$$0 = A - \alpha B + \alpha^2 C,$$

uae cum habeat duas radices, utramlibet assumere licet; eritque $A' = B$ - $B' = C$. Perventum est ergo ad hanc aequationem differentialem p radus:

$$e^{-\alpha x}\int e^{\alpha x}Xdx = A'y + \frac{B'dy}{dx}.$$

l quam denuo integrandam multiplicetur per $e^{\beta x}dx$, ut habeatur

$$e^{(\beta-\alpha)x}dx\int e^{\alpha x}Xdx = A'e^{\beta x}ydx + B'e^{\beta x}dy,$$

ae ut sit integrabilis, debet esse

$$\beta = \frac{A'}{B'} = \frac{B - \alpha C}{C} \quad \text{seu} \quad \alpha + \beta = \frac{B}{C},$$

de patet β esse alteram radicem aequationis

$$0 = A - \alpha B + \alpha^2 C,$$

tque integrale:

$$\int e^{(\beta-\alpha)x}dx\int e^{\alpha x}Xdx = B'e^{\beta x}y = Ce^{\beta x}y.$$

t vero

$$\int e^{(\beta-\alpha)x}dx\int e^{\alpha x}Xdx = \frac{e^{(\beta-\alpha)x}}{\beta-\alpha}\int e^{\alpha x}Xdx - \frac{1}{\beta-\alpha}\int e^{\beta x}Xdx,$$

o

$$Cy = \frac{e^{-\alpha x}}{\beta-\alpha}\int e^{\alpha x}Xdx + \frac{e^{-\beta x}}{\alpha-\beta}\int e^{\beta x}Xdx.$$

hac aequatione integrali ambae radices α et β aequationis quadraticae

$$0 = A - Bz + Czz$$

qualiter insunt, et hanc ob rem si istius aequationis radices sint cogni iis statim aequatio integralis formatur. Ista autem aequatio

ex ipsa aequatione proposita

$$X = Ay + \frac{Bdy}{dx} + \frac{Cddy}{dx^2}$$

facillime formatur simili scilicet modo, quo in casu $X = 0$ su
enim

$$1 \text{ pro } y, \ z \text{ pro } \frac{dy}{dx} \text{ et } z^2 \text{ pro } \frac{ddy}{dx^2},$$

ut prodeat ista expressio $A + Bz + Czz$; cuius factores si fueri
erunt α et β eae ipsae litterae, quae ad aequationem integ
requiruntur.

9. His praemissis additus ad integrationem aequatio
adeo erit difficilis. Sit ergo proposita haec aequatio:

$$X = Ay + \frac{Bdy}{dx} + \frac{Cddy}{dx^2} + \frac{Dd^3y}{dx^3} + \frac{Ed^4y}{dx^4} + \text{et}$$

cuius ultimus terminus sit $\frac{\varDelta d^ny}{dx^n}$. Formetur hinc ista exp
indicato:

$$A + Bz + Cz^2 + Dz^3 + Ez^4 + \ldots + \varDelta z^n =$$

quae in factores simplices resoluta sit:

$$P = \varDelta (z + \alpha)(z + \beta)(z + \gamma)(z + \delta) \text{ etc}$$

Dico iam, si aequatio differentialis proposita per $e^{\alpha x}dx$ r
evadere integrabilem. Erit enim

$$e^{\alpha x}Xdx = e^{\alpha x}dx\Big(Ay + \frac{Bdy}{dx} + \frac{Cddy}{dx^2} + \frac{Dd^3y}{dx^3} + \ldots$$

cuius integrale ponamus esse:

$$\int e^{\alpha x}Xdx = e^{\alpha x}\Big(A'y + \frac{B'dy}{dx} + \frac{C'ddy}{dx^2} + \frac{D'd^3y}{dx^3} + \ldots$$

Sumto autem differentiali habebitur

$$e^{\alpha x}Xdx = e^{\alpha x}dx\Big(\alpha A'y + \frac{A'dy}{dx} + \frac{B'ddy}{dx^2} + \frac{C'd^3y}{dx^3} + \ldots$$

$$+ \frac{\alpha B'dy}{dx} + \frac{\alpha C'ddy}{dx^2} + \ldots,$$

$$A' = \frac{A}{\alpha}$$
$$B' = \frac{B}{\alpha} - \frac{A}{\alpha^2}$$
$$C' = \frac{C}{\alpha} - \frac{B}{\alpha^2} + \frac{A}{\alpha^3}$$
$$D' = \frac{D}{\alpha} - \frac{C}{\alpha^2} + \frac{B}{\alpha^3} - \frac{A}{\alpha^4}$$

valoribus usque ad ultimum continuatis, pervenietur ad hanc aequa-

$$A - B\alpha + C\alpha^2 - D\alpha^3 + E\alpha^4 - \ldots \pm \Delta\alpha^n = 0;$$

ur α sit radix huius aequationis, erit $z + \alpha$ factor istius expressionis

$$P = A + Bz + Cz^2 + Dz^3 + Ez^4 + \ldots + \Delta z^n,$$

e $P = \Delta(z + \alpha)(z + \beta)(z + \gamma)(z + \delta)$ etc.

Prima ergo integratione absoluta erit

$$e^{-\alpha x}\int e^{\alpha x}X dx = A'y + \frac{B'dy}{dx} + \frac{C'ddy}{dx^2} + \frac{D'd^3y}{dx^3} + \ldots + \frac{\Delta d^{n-1}y}{dx^{n-1}}.$$

ur hinc iterum modo ante exposito haec expressio:

$$P' = A' + B'z + C'z^2 + D'z^3 + \ldots + \Delta z^{n-1}.$$

m sit:

$$A = \alpha A'$$
$$B = \alpha B' + A'$$
$$C = \alpha C' + B'$$
$$D = \alpha D' + C'$$
etc.

tum est fore $P = (\alpha + z)P'$, ideoque

$$P' = \frac{P}{z + \alpha} \text{ et}$$
$$P' = \Delta(z + \beta)(z + \gamma)(z + \delta)(z + \varepsilon) \text{ etc.}$$

nili ergo modo, quo supra usi sumus, evincetur hanc aequationem denuo ntegrabilem, si multiplicetur per $e^{\beta x}dx$.

$$\int e^{(\beta-\alpha)x}dx\int e^{\alpha x}Xdx = e^{\beta x}\Big(A''y + \frac{B''dy}{dx} + \frac{C''ddy}{dx^2} + \dots$$

fietque comparatione instituta

$$\begin{aligned} A' &= \beta A'' \\ B' &= \beta B'' + A'' \\ C' &= \beta C'' + B'' \\ D' &= \beta D'' + C'' \\ &\text{etc.} \end{aligned}$$

Ergo si ponatur

$$P'' = A'' + B''z + C''z^2 + D''z^3 + \dots + \varDelta z$$

erit $P' = (\beta + z)P''$ et

$$P'' = \frac{P'}{z+\beta} = \frac{P}{(z+\alpha)(z+\beta)},$$

unde fit

$$P'' = \varDelta\,(z+\gamma)\,(z+\delta)\,(z+\varepsilon)\ \text{etc.},$$

scilicet hinc duo iam factores $z+\alpha$ et $z+\beta$ sunt egressi

$$\int e^{(\beta-\alpha)x}dx\int e^{\alpha x}Xdx = \frac{e^{(\beta-\alpha)x}}{\beta-\alpha}\int e^{\alpha x}Xdx - \frac{1}{\beta-\alpha}\int$$

unde aequatio bis integrata reducitur ad hanc formam

$$\frac{e^{-\alpha x}}{\beta-\alpha}\int e^{\alpha x}Xdx + \frac{e^{-\beta x}}{\alpha-\beta}\int e^{\beta x}Xdx = A''y + \frac{B''dy}{dx}$$

$$+ \frac{D''d^3y}{dx^3} + \dots + \frac{\varDelta d^{n-2}y}{dx^{n-2}}.$$

11. Cum porro hinc posito 1 pro y et z pro $\frac{dy}{dx}$ etc. pro

$$P'' = A'' + B''z + C''z^2 + \dots + \varDelta z^n$$

sitque $\qquad P'' = \varDelta\,(z+\gamma)\,(z+\delta)\,(z+\varepsilon)$ etc.,

manifestum est aequationem ultimo inventam denuo r multiplicetur per $e^{\gamma x}dx$. Sit aequatio integralis hinc oriu

$$\int\frac{e^{(\gamma-\alpha)x}dx}{\beta-\alpha}\int e^{\alpha x}Xdx + \int\frac{e^{(\gamma-\beta)x}dx}{\alpha-\beta}\int e^{\beta x}Xd$$

$$e^{\gamma x}\Big(A'''y + \frac{B'''dy}{dx} + \frac{C'''ddy}{dx^2} + \dots + \frac{\varDelta d}{d}$$

$$B'' = \gamma B''' + A'''$$
$$C'' = \gamma C''' + B'''$$
$$D'' = \gamma D''' + C'''$$
etc.

i ponatur:

$$P''' = A''' + B'''z + C'''z^2 + D'''z^3 + \ldots + \Delta z^{n-3},$$

$$P'' = (\gamma + z)P''' \quad \text{et} \quad P''' = \frac{P''}{z+\gamma} = \frac{P}{(z+\alpha)(z+\beta)(z+\gamma)},$$

quitur fore:

$$P''' = \Delta (z+\delta)(z+\varepsilon)(z+\zeta) \text{ etc.}$$

tem sit generaliter

$$\int e^{(\mu-\nu)x} dx \int e^{\nu x} X dx = \frac{e^{(\mu-\nu)x}}{\mu-\nu}\int e^{\nu x} X dx + \frac{1}{\nu-\mu}\int e^{\mu x} X dx,$$

integralia reducantur, reperietur:

$$\frac{e^{-\alpha x}}{(\beta-\alpha)(\gamma-\alpha)}\int e^{\alpha x} X dx + \frac{e^{-\beta x}}{(\alpha-\beta)(\gamma-\beta)}\int e^{\beta x} X dx + \frac{e^{-\gamma x}}{(\alpha-\gamma)(\beta-\gamma)}\int e^{\gamma x} X dx$$
$$= A'''y + \frac{B'''dy}{dx} + \frac{C'''ddy}{dx^2} + \frac{D'''d^3y}{dx^3} + \ldots + \frac{\Delta d^{n-3}y}{dx^{n-3}}.$$

Si hoc modo eo usque progrediamur, quoad nulla amplius differensius y supersint, tum ex altera parte aequationis habebitur unicus ter- $\frac{d^0y}{dx^0} = \Delta y$; id quod eveniet, si integratio toties fuerit instituta, quot s exponens n continet unitates. Ad hoc ergo ultimum integrale comxprimendum, cum sit

$$+ Bz + Cz^2 + Dz^3 + \ldots + \Delta z^n = \Delta (z+\alpha)(z+\beta)(z+\gamma) \text{ etc.},$$

ur ex radicibus $\alpha, \beta, \gamma, \delta$ etc. sequentes valores

$$\mathfrak{A} = \Delta (\beta-\alpha)(\gamma-\alpha)(\delta-\alpha)(\varepsilon-\alpha) \text{ etc.}$$
$$\mathfrak{B} = \Delta (\alpha-\beta)(\gamma-\beta)(\delta-\beta)(\varepsilon-\beta) \text{ etc.}$$
$$\mathfrak{C} = \Delta (\alpha-\gamma)(\beta-\gamma)(\delta-\gamma)(\varepsilon-\gamma) \text{ etc.}$$
$$\mathfrak{D} = \Delta (\alpha-\delta)(\beta-\delta)(\gamma-\delta)(\varepsilon-\delta) \text{ etc.}$$
$$\mathfrak{E} = \Delta (\alpha-\varepsilon)(\beta-\varepsilon)(\gamma-\varepsilon)(\delta-\varepsilon) \text{ etc.}$$
etc.,

$$y = \frac{1}{\mathfrak{A}}\int e^{\alpha x} X dx + \frac{1}{\mathfrak{B}}\int e^{\beta x} X dx + \frac{1}{\mathfrak{C}}\int e^{\gamma x} X dx + \text{etc.},$$

quae cum tot contineat terminos, quoti gradus fuerit aequatio d
proposita

$$X = Ay + \frac{Bdy}{dx} + \frac{Cddy}{dx^2} + \frac{Dd^3y}{dx^3} + \ldots\ldots + \frac{\varDelta d^n y}{dx^n},$$

totidem involvet constantes arbitrarias, ideoque erit integralis com

13. Alio autem modo valores quantitatum $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$ etc. expri
qui plerumque multo commodius negotium conficit. Dico enim fo
si ubique pro z substituatur $-\alpha$, seu si ponatur $z + \alpha = 0$. Cum e

$$P = \varDelta (z + \alpha)(z + \beta)(z + \gamma)(z + \delta) \text{ etc.},$$

erit differentiando:

$$\frac{dP}{dz} = \varDelta(z + \beta)(z + \gamma)(z + \delta) \text{ etc.} + \frac{\varDelta(z + \alpha)}{dz} d \cdot (z + \beta)(z + \gamma)(z -$$

Si iam ponatur $z = -\alpha$, posterius membrum evanescet, et prius

$$\frac{dP}{dz} = \varDelta(\beta - \alpha)(\gamma - \alpha)(\delta - \alpha) \text{ etc.} = \mathfrak{A}.$$

Cum autem sit $P = A + Bz + Cz^2 + Dz^3 + \ldots. + \varDelta z^n$, erit

$$\frac{dP}{dz} = B + 2Cz + 3Dz^2 + 4Ez^3 + \ldots. + n\varDelta z^{n-1};$$

ponatur ergo $z = -\alpha$, seu fiat $z + \alpha = 0$, erit

$$\mathfrak{A} = B - 2C\alpha + 3D\alpha^2 - 4E\alpha^3 + \text{ etc. } \ldots. \pm n\varDelta \alpha^{n-1}$$

simili modo reperietur fore

$$\mathfrak{B} = B - 2C\beta + 3D\beta^2 - 4E\beta^3 + \ldots. \pm n\varDelta\beta^{n-1}$$
$$\mathfrak{C} = B - 2C\gamma + 3D\gamma^2 - 4E\gamma^3 + \ldots. \pm n\varDelta\gamma^{n-1}$$
$$\text{etc.}$$

$$X = Ay + \frac{}{dx} + \frac{}{dx^2} + \frac{}{dx^3} + \frac{}{dx^4} + \text{etc.},$$

am integrari oporteat, ante omnia ex ea formetur haec expressio Algebr

$$P = A + Bz + Cz^2 + Dz^3 + Ez^4 + \text{etc.},$$

us quaerantur omnes factores simplices, cuiusmodi unus sit $z + \alpha$, a
libet factor dabit partem integralis ita, ut omnes partes, quae hoc mod
gulis factoribus eruuntur, iunctim sumtae exhibeant completum ipsi
orem finitum. Scilicet si factor simplex fuerit inventus $z + \alpha$, tum quaer
antitas $\mathfrak{A}$, ut sit

$$\mathfrak{A} = B - 2C\alpha + 3D\alpha^2 - 4E\alpha^3 + \text{etc.},$$

a inventa erit pars integralis ex hoc factore $z + \alpha$ oriunda haec

$$\frac{e^{-\alpha x}}{\mathfrak{A}} \int e^{\alpha x} X dx.$$

nc perspicitur, si factor simplex formae P fuerit $z - \alpha$, tum fore

$$\mathfrak{A} = B + 2C\alpha + 3D\alpha^2 + 4E\alpha^3 + \text{etc.}$$

que integralis partem hinc oriundam esse[1])

$$\frac{e^{\alpha x}}{\mathfrak{A}} \int e^{-\alpha x} X dx.$$

15. Superest autem ut ostendamus, quomodo istae integralis partes
mparatae, si factorum simplicium aliquot fuerint vel inter se aequale
aginarii. Ex superioribus enim liquet utroque casu partes integralis s
i modo adornari debere, ut formam finitam et realem obtineant. Sint i
imo duo factores $z - \alpha$ et $z - \beta$ inter se aequales seu $\beta = \alpha$, eritque
$= 0$ quam $\mathfrak{B} = 0$; et utraque pars integralis evadet infinita, altera qu
firmative altera negative, ita ut differentia sit finita. Ad quam invenien
onamus $\beta = \alpha + \omega$, denotante ω quantitatem evanescentem. Cum erg

1) Cf. Commentationes 679, 680, 720 voluminis I 23.

sumtis litteris $\alpha, \beta, \gamma, \delta$ etc. negativis, erit¹)

$$\mathfrak{A} = -\varDelta\omega(\alpha-\gamma)(\alpha-\delta)(\alpha-\varepsilon) \text{ etc. et}$$
$$\mathfrak{B} = \varDelta\omega(\alpha-\gamma)(\alpha-\delta)(\alpha-\varepsilon) \text{ etc.}$$

Tum vero erit

$$e^{\beta x} = e^{\alpha x+\omega x} = e^{\alpha x}(1+\omega x) \text{ et } e^{-\beta x} = e^{-\alpha x}(1$$

Hinc pars integralis ex factoribus binis aequalibus $z-\alpha$ e

$$\frac{e^{\alpha x}}{\mathfrak{A}}\int e^{-\alpha x}Xdx + \frac{e^{\alpha x}(1+\omega x)}{\mathfrak{B}}\int e^{-\alpha x}(1-\omega x)X$$

Ponatur:

$$\mathfrak{A}' = \varDelta(\alpha-\gamma)(\alpha-\delta)(\alpha-\varepsilon) \text{ etc.}$$

erit $\mathfrak{A} = -\mathfrak{A}'\omega$ et $\mathfrak{B} = \mathfrak{A}'\omega$,

unde fiet ista pars

$$= \frac{e^{\alpha x}}{\mathfrak{A}'\omega}((1+\omega x)\int e^{-\alpha x}(1-\omega x)Xdx - \int e^{-\alpha x}X$$
$$= \frac{e^{\alpha x}}{\mathfrak{A}'\omega}(\omega x\int e^{-\alpha x}Xdx - \omega\int e^{-\alpha x}Xxdx)$$
$$= \frac{e^{\alpha x}}{\mathfrak{A}'}(x\int e^{-\alpha x}Xdx - \int e^{-\alpha x}Xxdx) = \frac{e^{\alpha x}}{\mathfrak{A}'}\int dx\int e$$

quae est pars integralis ex factore expressionis P quadr

16. Valor autem ipsius $\mathfrak{A}'$ sequenti modo commod
Ob $\beta = \alpha$, cum sit

$$P = A(z-\alpha)^2(z-\gamma)(z-\delta)(z-\varepsilon) \text{ etc.} = A + Bz +$$

ponatur

$$\varDelta(z-\gamma)(z-\delta)(z-\varepsilon) \text{ etc.} = Q,$$

ita ut valor ipsius Q praebeat $\mathfrak{A}'$ si loco z ponatur α. E

$$P = (z-\alpha)^2 Q,$$

et differentiando

---

1) Solutio sequens est vitiosa, quia omissum est ω in $(\beta-\gamma)(\beta-\delta$
tutionum calculi integralis volumine II notas ipsius EULERI, § 1163—1179.
Solutionem exactam attulit nota p. 330. *LEONHARDI EULERI Opera omnia*,

$$\frac{ddP}{dz^2} = (z-\alpha)^2 \frac{ddQ}{dz^2} + 4(z-\alpha)\frac{dQ}{dz} + 2Q;$$

unc $z = \alpha$ fiet

$$Q = \frac{ddP}{2\,dz^2} = \mathfrak{A}',$$

ue $\mathfrak{A}'$, si in $\frac{ddP}{2\,dz^2}$ ponatur $z = \alpha$. Est vero

$$\frac{ddP}{2dz^2} = C + 3Dz + 6Ez^2 + 10Fz^3 + 15Gz^4 + \text{etc.},$$

$$\mathfrak{A}' = C + 3D\alpha + 6E\alpha^2 + 10F\alpha^3 + 15G\alpha^4 + \text{etc.}$$

i proposita hac aequatione:

$$X = Ay + \frac{Bdy}{dx} + \frac{Cddy}{dx^2} + \frac{Dd^3y}{dx^3} + \frac{Ed^4y}{dx^4} + \text{etc.}$$

o hinc formata

$$P = A + Bz + Cz^2 + Dz^3 + Ez^4 + \text{etc.}$$

factorem quadratum $(z-\alpha)^2$, sumatur

$$\mathfrak{A}' = C + 3D\alpha + 6E\alpha^2 + 10F\alpha^3 + 15G\alpha^4 + \text{etc.}$$

pars integralis inde oriunda:

$$\frac{e^{\alpha x}}{\mathfrak{A}'}\int dx \int e^{-\alpha x}X dx.$$

em reliqui factores formulae P fuerint cogniti, nempe

$$P = \Delta(z-\alpha)^2(z-\gamma)(z-\delta)(z-\varepsilon) \text{ etc., erit}$$
$$\mathfrak{A}' = \Delta(\alpha-\gamma)(\alpha-\delta)(\alpha-\varepsilon) \text{ etc.}$$

Ponamus iam tres factores inter se esse aequales, seu sit insuper
at ob rationes supra expositas ponamus $\gamma = \alpha + \omega$, erit

$$\mathfrak{A}' = -\Delta\,\omega(\alpha-\delta)(\alpha-\varepsilon)(\alpha-\zeta) \text{ etc. et}$$
$$\mathfrak{C} = \Delta(\gamma-\alpha)^2(\gamma-\delta)(\gamma-\varepsilon)(\gamma-\zeta) \text{ etc. seu}$$

cubico $(z-\alpha)^3$ oriunda haec

$$\frac{e^{\alpha x}}{\mathfrak{A}''}\int dx\int dx\int e^{-\alpha x}Xdx$$

existente:

$$\mathfrak{A}'' = D + 4\,E\alpha + 10\,F\alpha^2 + 20\,G\alpha^3$$

Facilius autem hoc immediate ex aequalitate trium fa
enim tres factores quicunque $(z-\alpha)\,(z-\beta)\,(z-\gamma)$ ac

$$\mathfrak{A} = \Delta\,(\alpha-\beta)\,(\alpha-\gamma)\,(\alpha-\delta)\,(\alpha-\varepsilon)$$
$$\mathfrak{B} = \Delta\,(\beta-\alpha)\,(\beta-\gamma)\,(\beta-\delta)\,(\beta-\varepsilon)$$
$$\mathfrak{C} = \Delta\,(\gamma-\alpha)\,(\gamma-\beta)\,(\gamma-\delta)\,(\gamma-\varepsilon)$$

erunt integralis partes hinc oriundae:

$$\frac{e^{\alpha x}}{\mathfrak{A}}\int e^{-\alpha x}Xdx + \frac{e^{\beta x}}{\mathfrak{B}}\int e^{-\beta x}Xdx + \frac{e^{\gamma x}}{\mathfrak{C}}\int e$$

Ponatur iam

$$\beta = \alpha + \omega \text{ et } \gamma = \alpha + \Phi,$$

existentibus ω et Φ quantitatibus evanescentibus,

$$\mathfrak{A}'' = \Delta\,(\alpha-\delta)\,(\alpha-\varepsilon)\,(\alpha-\zeta) \text{ etc.}$$

erit $\quad \mathfrak{A} = \mathfrak{A}''\omega\Phi,\ \mathfrak{B} = \mathfrak{A}''\omega\,(\omega-\Phi)$ et $\mathfrak{C} = \mathfrak{A}''$

tum vero erit

$$e^{\beta x} = e^{\alpha x}\,(1 + \omega x + \tfrac{1}{2}\,\omega^2x^2),\ e^{-\beta x} = e^{-\alpha x}\,(1 -$$

et

$$e^{\gamma x} = e^{\alpha x}\,(1 + \Phi x + \tfrac{1}{2}\Phi^2x^2),\ e^{-\gamma x} = e^{-\alpha x}\,(1 -$$

Quibus substitutis ternae integralis partes abeunt i

$$\frac{e^{\alpha x}}{\mathfrak{A}''\omega\Phi(\omega-\Phi)}\begin{cases}\int e^{-\alpha x}Xdx\,(\omega - \Phi + \Phi + \omega\Phi x + \tfrac{1}{2}\omega^2\Phi x^2 - \\ \int e^{-\alpha x}Xxdx\,(-\omega\Phi - \omega\omega\Phi x + \omega\Phi + \omega \\ \int e^{-\alpha x}Xx^2\,dx\,(\tfrac{1}{2}\omega\omega\Phi - \tfrac{1}{2}\omega\Phi\Phi)\end{cases}$$

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1) Vide notam p. 202 huius voluminis.

$$\frac{e^{\alpha x}}{\mathfrak{A}''}(\tfrac{1}{2}xx\int e^{-\alpha x}Xdx - x\int e^{-\alpha x}Xxdx + \tfrac{1}{2}\int e^{-\alpha x}Xxxdx),$$

educitur ad hanc formam simpliciorem:

$$\frac{e^{\alpha x}}{\mathfrak{A}''}\int dx\int dx\int e^{-\alpha x}Xdx,$$

to $\mathfrak{A}'' = D + 4\,Ea + 10\,Fa^2 + 20\,Ga^3 +$ etc., scilicet valor ipsius $\mathfrak{A}''$ ex formula $\frac{d^3P}{6\,dz^3}$ posito $z = a$.

Simili modo ulterius procedendo patebit quaternos factores inter ales seu formulae

$$P = A + Bz + Cz^2 + \text{ etc.}$$

n $(z-a)^4$ praebiturum fore hanc integralis partem[1]):

$$\frac{e^{\alpha x}\int dx\int dx\int dx\int e^{-\alpha x}Xdx}{E + 5Fa + 15Ga^2 + 35Ha^3 + \text{etc.}},$$

ominator ex formula $\frac{d^4P}{24\,dz^4}$ nascitur ponendo $z = a$. Superfluum foret ribus factoribus simplicibus inter se aequalibus partes integralis, quae conflantur, hic exhibere, cum lex, qua hae partes formantur, per se ifesta. Ceterum complicatio plurium signorum integralium in his formullam involvit difficultatem, cum facillime ad simplicia integralia tur. Est enim

$$\int dx\int e^{-\alpha x}Xdx = \frac{x\int e^{-\alpha x}Xdx - \int e^{-\alpha x}Xxdx}{1}$$

$$\int dx\int dx\int e^{-\alpha x}Xdx = \frac{x^2\int e^{-\alpha x}Xdx - 2x\int e^{-\alpha x}Xxdx + \int e^{-\alpha x}Xxxdx}{1.\ 2}$$

$$dx\int e^{-\alpha x}Xdx = \frac{x^3\int e^{-\alpha x}Xdx - 3x^2\int e^{-\alpha x}Xxdx + 3x\int e^{-\alpha x}Xxxdx - \int e^{-\alpha x}Xx^3dx}{1.\ 2.\ 3}$$

etc.

Expeditis factoribus aequalibus pergo ad factores imaginarios. Sint mulae

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[1]) ide notam p. 202 huius voluminis. H. D.

bini factores $z-\alpha$ et $z-\beta$ imaginarii, qui hoc non obstante habeant productum reale $zz - 2kz\cos.\Phi + kk$; erit ergo

$$\alpha = k\cos.\Phi + k\sqrt{-1}\sin.\Phi \text{ et}$$
$$\beta = k\cos.\Phi - k\sqrt{-1}\sin.\Phi$$

harumque litterarum potestates quaecunque ita se habebunt

$$\alpha^n = k^n\cos.n\Phi + k^n\sqrt{-1}\sin.n\Phi$$
$$\beta^n = k^n\cos.n\Phi - k^n\sqrt{-1}\sin.n\Phi$$

Iam primo erit[1]):

$$e^{\alpha x} = e^{kx\cos.\Phi}\Big(1 + \frac{k\sqrt{-1}}{1}x\sin.\Phi - \frac{kk}{1\cdot 2}x^2\sin.\Phi^2 \ldots$$
$$+ \frac{k^4}{1\cdot 2\cdot 3\cdot 4}x^4\sin.\Phi^4 \ldots \text{etc.}\Big)$$

ideoque

$$e^{\alpha x} = e^{kx\cos.\Phi}(\cos.kx\sin.\Phi + \sqrt{-1}\sin. \ldots$$
$$e^{\beta x} = e^{kx\cos.\Phi}(\cos.kx\sin.\Phi - \sqrt{-1}\sin. \ldots$$
$$e^{-\alpha x} = e^{-kx\cos.\Phi}(\cos.kx\sin.\Phi - \sqrt{-1}\sin. \ldots$$
$$e^{-\beta x} = e^{-kx\cos.\Phi}(\cos.kx\sin.\Phi + \sqrt{-1}\sin. \ldots$$

Deinde cum sit:

$$\mathfrak{A} = B + 2C\alpha + 3D\alpha^2 + 4E\alpha^3 + 5F\alpha^4 + \ldots$$
$$\mathfrak{B} = B + 2C\beta + 3D\beta^2 + 4E\beta^3 + 5F\beta^4 + \ldots$$

superioribus valoribus pro α et β substitutis habebitur

$$\mathfrak{A} = \begin{array}{l} B + 2Ck\cos.\Phi + 3Dk^2\cos.2\Phi + 4Ek^3\cos. \ldots \\ + (2Ck\sin.\Phi + 3Dk^2\sin.2\Phi + 4Ek^3\sin. \ldots \end{array}$$

$$\mathfrak{B} = \begin{array}{l} B + 2Ck\cos.\Phi + 3Dk^2\cos.2\Phi + 4Ek^3\cos. \ldots \\ - (2Ck\sin.\Phi + 3Dk^2\sin.2\Phi + 4Ek^3\sin. \ldots \end{array}$$

20. Cum autem $z-\alpha$ et $z-\beta$ sint factores formulae

erit $$P = A + Bz + Cz^2 + Dz^3 + Ez^4 + \ldots$$

$$A + Bk\cos.\Phi + Ck^2\cos.2\Phi + Dk^3\cos.3\Phi + Ek^4 \ldots$$

et $$Bk\sin.\Phi + Ck^2\sin.2\Phi + Dk^3\sin.3\Phi + Ek^4 \ldots$$

---

[1]) $\sin.\Phi^n = (\sin\Phi)^n$.

$$\mathfrak{N} = \qquad 2Ck \sin.\Phi + 3Dk^2 \sin.2\Phi + 4Ek^3 \sin.3\Phi + \text{ etc.}$$

et:

$$\mathfrak{A} = \mathfrak{M} + \mathfrak{N}\sqrt{-1} \text{ et } \mathfrak{B} = \mathfrak{M} - \mathfrak{N}\sqrt{-1}$$

naginaria a realibus erunt separata. Cum nunc ex ambobus factoribus $z - \beta$ nascantur istae integralis partes

$$\frac{e^{\alpha x}}{\mathfrak{A}}\int e^{-\alpha x} X dx + \frac{e^{-\beta x}}{\mathfrak{B}}\int e^{-\beta x} X dx,$$

ount in hanc formam:

$$\frac{(\mathfrak{M} - \mathfrak{N}\sqrt{-1})\, e^{\alpha x}\int e^{-\alpha x} X dx + (\mathfrak{M} + \mathfrak{N}\sqrt{-1})\, e^{\beta x}\int e^{-\beta x} X dx}{\mathfrak{M}^2 + \mathfrak{N}^2}.$$

$$e^{-\alpha x} X dx = \begin{cases} + e^{kx\cos.\Phi}\cos.kx\sin.\Phi \int e^{-kx\cos.\Phi} X dx \cos.kx\sin.\Phi \\ - \sqrt{-1}\cdot e^{kx\cos.\Phi}\cos.kx\sin.\Phi \int e^{-kx\cos.\Phi} X dx \sin.kx\sin.\Phi \\ + \sqrt{-1}\cdot e^{kx\cos.\Phi}\sin.kx\sin.\Phi \int e^{-kx\cos.\Phi} X dx \cos.kx\sin.\Phi \\ + e^{kx\cos.\Phi}\sin.kx\sin.\Phi \int e^{-kx\cos.\Phi} X dx \sin.kx\sin.\Phi \end{cases}$$

$$e^{-\beta x} X dx = \begin{cases} + e^{kx\cos.\Phi}\cos.kx\sin.\Phi \int e^{-kx\cos.\Phi} X dx \cos.kx\sin.\Phi \\ + \sqrt{-1}\cdot e^{kx\cos.\Phi}\cos.kx\sin.\Phi \int e^{-kx\cos.\Phi} X dx \sin.kx\sin.\Phi \\ - \sqrt{-1}\cdot e^{kx\cos.\Phi}\sin.kx\sin.\Phi \int e^{-kx\cos.\Phi} X dx \cos.kx\sin.\Phi \\ + e^{kx\cos.\Phi}\sin.kx\sin.\Phi \int e^{-kx\cos.\Phi} X dx \sin.kx\sin.\Phi. \end{cases}$$

rgo ambae integrales transibunt, imaginariis se mutuo sublatis, in hanc

$$\frac{\ldots^{\cos.\Phi}}{\ldots\mathfrak{N}^2}\Big(\cos.kx\sin.\Phi \int e^{-kx\cos.\Phi} X dx \cos.kx\sin.\Phi$$
$$+ \sin.kx\sin.\Phi \int e^{-kx\cos.\Phi} X dx \sin.kx\sin.\Phi$$

$$\frac{\ldots^{\cos.\Phi}}{\ldots\mathfrak{N}^2}\Big(\sin.kx\sin.\Phi \int e^{-kx\cos.\Phi} X dx \cos.kx\sin.\Phi$$
$$- \cos.kx\sin.\Phi \int e^{-kx\cos.\Phi} X dx \sin.kx\sin.\Phi\Big)$$

iam hoc modo exprimi potest:

$$\begin{cases} \quad \mathfrak{M}\cos.kx\sin.\Phi + \mathfrak{N}\sin.kx\sin.\Phi)\int e^{-kx\cos.\Phi} X dx \cos.kx\sin.\Phi \\ + \mathfrak{M}\sin.kx\sin.\Phi - \mathfrak{N}\cos.kx\sin.\Phi)\int e^{-kx\cos.\Phi} X dx \sin.kx\sin.\Phi. \end{cases}$$

rgo pars integralis oritur ex formulae

$$P = A + Bz + Cz^2 + Dz^3 + \text{ etc.}$$

trinomiali $zz - 2kz\cos.\Phi + kk$.

factorem habuerit $(zz - 2kz \cos. \Phi + kk)^2$, pars integ
formulis pro binis factoribus simplicibus aequalibus su
Ponatur nempe

$$\mathfrak{M}' = C + 3Dk \cos. \Phi + 6Ek^2 \cos. 2\Phi + 10Fk$$
$$\mathfrak{N}' = \quad 3Dk \sin. \Phi + 6Ek^2 \sin. 2\Phi + 10Fk$$

eritque integralis pars hinc oriunda[1]),

$$\frac{2e^{kx\cos.\Phi}}{\mathfrak{M}'\mathfrak{M}' + \mathfrak{N}'\mathfrak{N}'} \left\{ \begin{array}{l} (\mathfrak{M}'\cos. kx \sin.\Phi + \mathfrak{N}'\sin. kx \sin.\Phi)\int dx \int e \\ + (\mathfrak{M}'\sin. kx \sin.\Phi - \mathfrak{N}'\cos. kx \sin.\Phi)\int dx \int e \end{array} \right.$$

Sin autem tres factores trinomiales radices imaginar
inter se aequales, seu si formulae

$$P = A + Bz + Cz^2 + Dz^3 + Ez^4 + Fz$$

factor fuerit $(zz - 2kz \cos. \Phi + kk)^3$, statuatur

$$\mathfrak{M}'' = D + 4Ek \cos. \Phi + 10Fk^2 \cos. 2\Phi + 20Gk$$
$$\mathfrak{N}'' = \quad 4Ek \sin. \Phi + 10Fk^2 \sin. 2\Phi + 20Gk$$

atque pars integralis ex hoc factore oriunda erit

$$\frac{2e^{kx\cos.\Phi}}{\mathfrak{M}''\mathfrak{M}'' + \mathfrak{N}''\mathfrak{N}''} \left\{ \begin{array}{l} (\mathfrak{M}''\cos. kx \sin.\Phi + \mathfrak{N}''\sin. kx \sin.\Phi)\int dx \int d \\ + (\mathfrak{M}''\sin. kx \sin.\Phi - \mathfrak{N}''\cos. kx \sin.\Phi)\int dx \int d \end{array} \right.$$

Hinc igitur iam lex perspicitur, secundum quam istae i
debent, si maior potestas formulae $zz - 2kz \cos. \Phi + k$
ideoque omnes casus, qui unquam occurrere possunt, l

22. Ex his ergo sequenti modo resolvi poterit hoc

PROBLEMA

Invenire valorem ipsius y in quantitatibus finitis ex
nit ex hac aequatione differentiali cuiuscunque gradu

1) Vide notas p. 3 et p. 202 huius voluminis adiectas. Confer quo
gralis, vol. II, § 1170—1184; LEONHARDI EULERI *Opera omnia*, I 12.

ubi differentiale dx ponitur constans, atque X denotat functionem quam-
ipsius x.

Solutio

Ex aequatione proposita formetur sequens formula Algebraica:

$$P = A + Bz + Cz^2 + Dz^3 + Ez^4 + Fz^5 + \text{ etc.},$$

cuius quaerantur omnes factores reales tam simplices, quam trino-
quippe qui factorum simplicium imaginariorum vices sustinent; et
quorum factorum inter se fuerint aequales, ii coniunctim repraesententu-
facto pro singulis factoribus quaerantur convenientes integralis partes
omnes istae partes ex cunctis factoribus ortundae, si in unam summa-
gantur, dabunt valorem ipsius y quaesitum, qui erit integrale com-
aequationis propositae. Sequenti autem modo ex factoribus formulae
gralis partes reperientur[1]).

I. *Si formulae P factor sit $z-k$*

Ponatur $\mathfrak{K} = B + 2Ck + 3Dk^2 + 4Ek^3 + 5Fk^4 + \text{etc.}$

eritque integralis pars huic factori $z-k$ respondens:

$$\frac{e^{kx}}{\mathfrak{K}}\int e^{-kx}Xdx.$$

II. *Si formulae P factor sit $(z-k)^2$*

Ponatur $\mathfrak{K} = C + 3Dk + 6Ek^2 + 10Fk^3 + 15Gk^4 + \text{ etc.}$

eritque integralis pars factori $(z-k)^2$ respondens:

$$\frac{e^{kx}}{\mathfrak{K}}\int dx\int e^{-kx}Xdx.$$

III. *Si formulae P factor sit $(z-k)^3$*

Ponatur $\mathfrak{K} = D + 4Ek + 10Fk^2 + 20Gk^3 + 35Hk^4 + \text{ etc.}$

eritque integralis pars factori $(z-k)^3$ respondens:

$$\frac{e^{kx}}{\mathfrak{K}}\int dx\int dx\int e^{-kx}Xdx.$$

---

1) Omnes hae formulae, exceptis I et V, sunt vitiosae. Vide notam p. 202 huius volumi-

Ponatur $\mathfrak{K} = E + 5Fk + 15Gk^2 + 35Hk^3 + 70Ik^4$
eritque integralis pars factori $(z-k)^4$ respondens:

$$\frac{e^{kx}}{\mathfrak{K}} \int dx \int dx \int dx \int e^{-kx} X dx.$$

V. *Si formulae P factor sit* $zz - 2kz$ co

Ponatur

$$\mathfrak{M} = B + 2Ck \cos.\phi + 3Dk^2 \cos.2\phi + 4Ek$$
$$\mathfrak{N} = \quad 2Ck \sin.\phi + 3Dk^2 \sin.2\phi + 4Ek$$

erit pars integralis factori $zz - 2kz \cos.\phi + kk$ resp

$$\frac{2e^{kx\cos.\phi}}{\mathfrak{M}^2 + \mathfrak{N}^2} \left\{ \begin{array}{l} (\mathfrak{M} \cos.kx \sin.\phi + \mathfrak{N} \sin.kx \sin.\phi) \int e^{-k} \\ + (\mathfrak{M} \sin.kx \sin.\phi - \mathfrak{N} \cos.kx \sin.\phi) \int e^{-k} \end{array} \right.$$

VI. *Si formulae P factor sit* $(zz - 2kz$ c

Ponatur

$$\mathfrak{M} = C + 3Dk \cos.\phi + 6Ek^2 \cos.2\phi + 10F$$
$$\mathfrak{N} = \quad 3Dk \sin.\phi + 6Ek^2 \sin.2\phi + 10F$$

erit pars integralis factori $(zz - 2kz \cos.\phi + kk)^2$ r

$$\frac{2e^{kx\cos.\phi}}{\mathfrak{M}^2 + \mathfrak{N}^2} \left\{ \begin{array}{l} (\mathfrak{M} \cos.kx \sin.\phi + \mathfrak{N} \sin.kx \sin.\phi) \int dx \int e^{-} \\ + (\mathfrak{M} \sin.kx \sin.\phi - \mathfrak{N} \cos.kx \sin.\phi) \int dx \int e^{-} \end{array} \right.$$

VII. *Si formulae P factor sit* $(zz - 2kz$ c

Ponatur

$$\mathfrak{M} = D + 4Ek \cos.\phi + 10Fk^2 \cos.2\phi + 20$$
$$\mathfrak{N} = \quad 4Ek \sin.\phi + 10Fk^2 \sin.2\phi + 20$$

erit pars integralis factori $(zz - 2kz \cos.\phi + kk)^3$ r

$$\frac{2e^{kx\cos.\phi}}{\mathfrak{M}^2 + \mathfrak{N}^2} \left\{ \begin{array}{l} (\mathfrak{M} \cos.kx \sin.\phi + \mathfrak{N} \sin.kx \sin.\phi) \int dx \int dx \int e \\ + (\mathfrak{M} \sin.kx \sin.\phi - \mathfrak{N} \cos.kx \sin.\phi) \int dx \int dx \int e \end{array} \right.$$

etc.

Omnes igitur istae partes singulis factoribus formulae
summam collectae dabunt valorem ipsius y quaesit

regulae huius usus facilius perspicietur.

Exemplum 1. Proposita sit haec aequatio differentialis secundi gradus:

$$X = y - \frac{ddy}{dx^2}.$$

igitur formula Algebraica P erit $= 1 - zz$, cuius factores sunt $z + 1$ et ex formula prima erit

$$\mathfrak{R} = \frac{dP}{dz} = -2z.$$

factore ergo $z + 1$ ob $k = -1$ erit $\mathfrak{R} = 2$ et pars integralis

$$= \frac{e^{-x}}{2}\int e^x X dx.$$

altero factore est $k = 1$ et $\mathfrak{R} = -2$, cui respondet pars integralis

$$-\frac{e^x}{2}\int e^{-x} X dx,$$

is partibus collectis erit integrale quaesitum

$$y = \tfrac{1}{2} e^{-x} \int e^x X dx - \tfrac{1}{2} e^x \int e^{-x} X dx.$$

Exemplum 2. Proposita sit haec aequatio:

$$X = y - \frac{3a dy}{dx} + \frac{3aa ddy}{dx^2} - \frac{a^3 d^3 y}{dx^3}.$$

ergo

$$P = 1 - 3az + 3aazz - a^3 z^3 = (1 - az)^3.$$

enda ergo est formula tertia, eritque

$$k = \frac{1}{a} \text{ et } \mathfrak{R} = \frac{d^3 P}{6 dz^3} = -a^3,$$

prodit integrale quaesitum

$$y = -\frac{1}{a^3} e^{x:a} \int dx \int dx \int e^{-x:a} X dx$$

$$y = -\frac{1}{a^3} e^{x:a} (x \int dx \int e^{-x:a} X dx - \int x dx \int e^{-x:a} X d$$

seu

$$y = -\frac{1}{a^3} e^{x:a} (\tfrac{1}{2} xx \int e^{-x:a} X dx - x \int e^{-x:a} X x dx + \tfrac{1}{2} \int e^{-x:}$$

Exemplum 3. Proposita sit haec aequatio:

$$X = y + \frac{aaddy}{dx^2}$$

Erit ergo $P = 1 + aazz$, quae ad formulam V pertinet. Erit

$$\cos.\,\Phi = 0,\ \sin.\,\Phi = 1 \text{ et } k = \frac{1}{a}.$$

Porro ob

$$A = 1,\ B = 0 \text{ et } C = aa, \text{ erit } \mathfrak{M} = 0, \text{ et } \mathfrak{N} = 2$$

unde erit integrale:

$$y = \frac{1}{a} \sin.\,\frac{x}{a} \int X dx \cos.\,\frac{x}{a} - \frac{1}{a} \cos.\,\frac{x}{a} \int X dx \sin.\,-$$

Exemplum 4. Proposita sit haec aequatio:

$$X = y + \frac{a^3 d^3 y}{dx^3}.$$

Erit ergo $P = 1 + a^3 z^3$, cuius duo sunt factores

$$1 + az \quad \text{et} \quad 1 - az + aazz,$$

Prior ad formam $z - k$ reductus, dat

$$k = -\frac{1}{a} \text{ et ob } A = 1,\ B = 0,\ C = 0 \text{ et } D = a$$

erit ex formula prima $\mathfrak{K} = 3a$ et pars integralis:

$$\frac{1}{3a} e^{-x:a} \int e^{x:a} X dx.$$

Alter factor

$$1 - az + aazz \quad \text{seu} \quad zz - \frac{z}{a} + \frac{1}{aa}$$

cum formula V comparatus dat

$$\mathfrak{M} = 3a \cos. 120^\circ = -\frac{3}{2}a \quad \text{et} \quad \mathfrak{N} = 3a \sin. 120^\circ = \frac{3a\sqrt{3}}{2},$$

$$\mathfrak{M}^2 + \mathfrak{N}^2 = 9aa \quad \text{atque} \quad \frac{2\mathfrak{M}}{\mathfrak{M}^2+\mathfrak{N}^2} = -\frac{1}{3a} \quad \text{et} \quad \frac{2\mathfrak{N}}{\mathfrak{M}^2+\mathfrak{N}^2} = \frac{\sqrt{3}}{3a}.$$

gralis ergo hinc oriunda est:

$$\frac{1}{3a} e^{x:2a}\left(-\cos. \frac{x\sqrt{3}}{2a} + \sqrt{3}\cdot\sin. \frac{x\sqrt{3}}{2a}\right)\int e^{-x:2a} X dx \cos. \frac{x\sqrt{3}}{2a}$$

$$+\frac{1}{3a} e^{x:2a}\left(-\sin. \frac{x\sqrt{3}}{2a} - \sqrt{3}\cdot\cos. \frac{x\sqrt{3}}{2a}\right)\int e^{-x:2a} X dx \sin. \frac{x\sqrt{3}}{2a}$$

$$-\frac{3a}{2} e^{x:2a} \cos.\left(\frac{x\sqrt{3}}{2a} + 60^\circ\right)\int e^{-x:2a} X dx \cos. \frac{x\sqrt{3}}{2a}$$

$$-\frac{2}{3a} e^{x:2a} \sin.\left(\frac{x\sqrt{3}}{2a} + 60^\circ\right)\int e^{-x:2a} X dx \sin. \frac{x\sqrt{3}}{2a}.$$

ur integrale quaesitum erit:

$$e^{-x:a}\int e^{x:a} X dx - \frac{2}{3a} e^{x:2a} \cos.\left(\frac{x\sqrt{3}}{2a} + 60^\circ\right)\int e^{-x:2a} X dx \cos. \frac{x\sqrt{3}}{2a}$$

$$-\frac{2}{3a} e^{x:2a} \sin.\left(\frac{x\sqrt{3}}{2a} + 60^\circ\right)\int e^{-x:2a} X dx \sin. \frac{x\sqrt{3}}{3a}.$$

o exempla sufficiunt ad regulam pro quovis casu oblato accommodan-

EXPOSITION DE QUELQUES PA
DANS LE CALCUL INTÉG

Commentatio 236 indicis ENESTROEMIANI

Mémoires de l'académie des sciences de Berlin 12 (1756),

PREMIER PARADOXE

I. Je me propose ici de développer un paradoxe
qui paroitra bien étrange: c'est qu'on parvient quelq
différentielles, dont il paroit fort difficile de trouver les
du calcul intégral, et qu'il est pourtant aisé de trouve
l'intégration, mais plutôt en différentiant encore l'équa
qu'une différentiation réiterée nous conduise dans ces ca
C'est sans doute un accident fort surprenant, que la dif
mener au même but, auquel on est accoutumé de parve
est une opération entierement opposée.

II. Pour mieux faire sentir l'importance de ce pa
souvenir, que le calcul intégral renferme la méthode
intégrales des quantités différentielles quelconques: et
équation différentielle étant proposée, il n'y a d'autre m
intégrale, que d'en entreprendre l'intégration. Et si l'o
tégrer cette équation, la différentier encore une fois,
s'éloigneroit encore davantage du but proposé; attend
équation différentielle du second degré, qu'il faudroit
avant qu'on parvint au but proposé.

rcher, mais qu'elle nous puisse même fournir cette intégrale. Ce seroit san un grand avantage, si cet accident étoit général, et qu'il eut lieu toujours r'alors la recherche des intégrales, qui est souvent même impossible oit plus la moindre difficulté: mais il ne se trouve qu'en quelques cas trè uliers dont je rapporterai quelques exemples: les autres cas demanden urs la méthode ordinaire d'intégration. Voilà donc quelques problèmes qu ont à éclaircir ce paradoxe.

PROBLEME I

e point A étant donné (Fig. 1), trouver la courbe EM telle, que la perpen ire AV tirée du point A sur une tangente quelconque de la courbe MV, so t de la même grandeur.

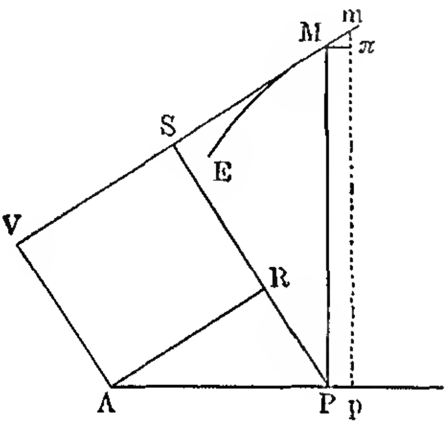

Fig. 1

V. Prenant pour axe une droite quelconque AP, tirée du point donné y tire d'un point quelconque de la courbe cherchée M la perpendiculair et une autre infiniment proche mp, et qu'on nomme $AP = x$, $PM =$ longueur donnée de la ligne $AV = a$. Soit de plus l'élément de la courb $= ds$, et ayant tiré $M\pi$ parallèle à l'axe AP, on aura

$$Pp = M\pi = dx \quad \text{et} \quad \pi m = dy;$$

$$ds = \sqrt{(dx^2 + dy^2)}.$$

n baisse du point P aussi sur la tangente MV la perpendiculaire PS, elle-cy du point A la perpendiculaire AR, qui sera parallèle à la tangen

MP. maintenant, par

triangle $Mm\pi$, on en tirera:

$$PS = \frac{M\pi \cdot PM}{Mm} = \frac{ydx}{ds} \quad \text{et} \quad PR = \frac{m\pi \cdot A}{Mm}$$

d'où, à cause de

$$AV = PS - PR,$$

nous aurons cette équation

$$a = \frac{ydx - xdy}{ds}$$

ou

$$ydx - xdy = ads = a\sqrt{(dx^2 + d}$$

qui exprimera la nature de la courbe cherchée.

V. Voila donc une équation différentielle pour la
chons: et si nous la voulons traiter selon la méthode
débarrasser les différentiels du signe radical; prenant
aurons:

$$yydx^2 - 2xydxdy + xxdy^2 = aadx^2$$

et partant

$$dy^2 = \frac{-2xydxdy - aadx^2 + yy}{aa - xx}$$

dont l'extraction de racine fournit

$$dy = \frac{-xydx + adx\sqrt{(xx + yy -}}{aa - xx}$$

ou

$$aady - xxdy + xydx = adx\sqrt{(xx +}$$

dont il faut maintenant chercher l'intégrale pour
question.

VI. Pour intégrer cette équation, posons $y = u$
pour avoir

$$\sqrt{(xx + yy - aa)} = \sqrt{(aa - xx)}\ (u$$

et

$$aady - xxdy = du\,(aa-xx)^{\frac{3}{2}} - uxdx\,\sqrt{(aa-xx)}.$$

rs étant substituées donnent:

$$du\,(aa-xx)^{\frac{3}{2}} = adx\,\sqrt{(aa-xx)}\,(uu-1)$$

$$\frac{du}{\sqrt{(uu-1)}} = \frac{adx}{aa-xx},$$

où les variables x et u se trouvent separées.

Puisque cette équation est séparée, je remarque d'abord, que les s, qu'elle renferme, sont remplies, si l'on met

$$\sqrt{(uu-1)} = 0, \quad \text{ou} \quad uu = 1;$$

ce cas tant le membre

$$adx\,\sqrt{(aa-xx)}\,(uu-1)$$

vanouïssant, que l'autre membre $du\,(aa-xx)^{\frac{3}{2}}$ à cause de $du = 0$. nt, nous avons déjà une valeur intégrale $uu = 1$, ou $u = \pm 1$, d'où ns $y = \pm\sqrt{(aa-xx)}$, ou $yy + xx = aa$; ce qui est l'équation pour , décrit du centre A avec le rayon $= a$. Or il est clair que ce cercle au problème, puisque la perpendiculaire AV devient égale au rayon , et tombe sur le point d'attouchement M; comme il est connu par les s du cercle.

. Mais ce cas n'épuise pas encore l'équation différentielle

$$\frac{du}{\sqrt{(uu-1)}} = \frac{adx}{aa-xx};$$

s donc son intégrale qui sera par les logarithmes

$$l\,(u+\sqrt{(uu-1)}) = \tfrac{1}{2}\,l\,\frac{nn\,(a+x)}{a-x},$$

que nous ayons :

$$u + \sqrt{(uu - 1)} = n \sqrt{\frac{a + x}{a - x}}.$$

De là nous trouverons,

$$-1 = nn \cdot \frac{a + x}{a - x} - 2nu \sqrt{\frac{a + x}{a - x}}$$

et partant

$$u = \frac{n}{2} \sqrt{\frac{a + x}{a - x}} + \frac{1}{2n} \sqrt{\frac{a - x}{a + x}}.$$

Par conséquent

$$y = u \sqrt{(aa - xx)} = \frac{n}{2}(a + x) + \frac{1}{2n} (a$$

équation pour une ligne droite tirée en sorte, que la perpen
sur elle du point donné A soit $= a$.

IX. Voilà donc la solution du problème proposé, qu'o
méthode ordinaire, où il faut premierement séparer les v
intégrer l'équation différentielle séparée. Or il est clair, que
non seulement assez embarrassante, mais elle deviendroit
si au lieu de la formule irrationnelle $\sqrt{(dx^2 + dy^2)}$, on en a
pliquée. Comme si l'on étoit parvenu à cette équation

$$ydx - xdy = a \sqrt[3]{(dx^3 + dy^3)},$$

en prenant des cubes, on auroit bien de la peine à extrair
pour trouver le rapport entre les différentiels dx et dy. Et si l
haute, cette extraction deviendroit même impossible.

X. Or maintenant je dis, que cette même équation qui
tion du problème $ydx - xdy = a\sqrt{(dx^2 + dy^2)}$ se peut rédu
finie, et même algébrique, entre x et y, sans y employer
d'intégration: mais, en quoi consiste la force du paradoxe,
tiation ultérieure de cette équation. Où ce sera cette même di
nous conduira à l'équation intégrale, qui nous fera connoîtr
courbe cherchée. Ce que je viens d'avancer, mettra dans tout
du paradoxe, que je me suis proposé de démêler ici.

$dx^2 + dy^2) = dx\sqrt{(1+pp)}$. Par cette substitution notre équation, é
visée par dx, prendra cette forme,

$$y - px = a\sqrt{(1+pp)} \text{ ou } y = px + a\sqrt{(1+pp)},$$

il faut bien remarquer, que quoiqu'on n'y apperçoive plus de différent
cette équation ne laisse pas d'être différentielle, à cause de la lettre p, dor
leur est $\frac{dy}{dx}$; de sorte que, si l'on la remettoit, on reviendroit à la prem
uation différentielle.

XII. A présent, au lieu d'intégrer cette équation différentielle, je la d
ntie encore une fois pour avoir

$$dy = pdx + xdp + \frac{apdp}{\sqrt{(1+pp)}}.$$

, ayant supposé $dy = pdx$, cette valeur mise à la place de dy nous do
abord:

$$0 = xdp + \frac{apdp}{\sqrt{(1+pp)}},$$

où en divisant par dp nous tirons d'abord:

$$x = -\frac{ap}{\sqrt{(1+pp)}}$$

puisqu'il y a

$$y = px + a\sqrt{(1+pp)},$$

y substituant cette valeur de

$$x = -\frac{ap}{\sqrt{(1+pp)}},$$

us aurons:

$$y = -\frac{app}{\sqrt{(1+pp)}} + a\sqrt{(1+pp)} \text{ ou } y = \frac{a}{\sqrt{(1+pp)}}.$$

XIII. Voilà donc des valeurs, et mêmes algébriques, pour les deux c
nnées x et y, lesquelles ne renferment que la seule variable p; et comm

présent il n'est plus question de la valeur supposée de $p =$
est résolu par cette différentiation réitérée. Car on n'a qu'à él
p de ces deux équations

$$x = -\frac{ap}{\sqrt{(1+pp)}} \quad \text{et} \quad y = \frac{a}{\sqrt{(1+pp)}},$$

ce qui se fera aisément en ajoutant ensemble les quarrés x
aura d'abord

$$xx + yy = \frac{aapp + aa}{1+pp} = aa,$$

qui est l'équation pour le cercle, qui satisfait au problème pr

XIV. Il est bien vray, qu'outre le cercle il y a encore un
droites, qui satisfont également à la question, et que cette m
pas fournir. Mais elle les contient néanmoins, et encore plus
l'autre méthode ordinaire. On n'a qu'à regarder l'équation

$$0 = xdp + \frac{apdp}{\sqrt{(1+pp)}},$$

à laquelle la différentiation nous a conduit, et qui, puisq
par dp, renferme aussi la solution $dp = 0$. Or de là nous tiron
$p = \text{const} = n$, et partant

$$y = nx + a\sqrt{(1+nn)},$$

où toutes les lignes droites, qui remplissent les conditions
comprises.

XV. Ayant déjà remarqué que cette équation:

$$ydx - xdy = a\sqrt[3]{(dx^3 + dy^3)}$$

ne sauroit à peine être résolue par la méthode ordinaire, cel
d'abord par la différentiation son intégrale. Car, posant $dy =$

$$\sqrt[3]{(dx^3 + dy^3)} = dx\sqrt[3]{(1+p^3)},$$

et partant

$$y - px = a\sqrt[3]{(1+p^3)} \text{ ou } y = px + a\sqrt[3]{(1 -$$

$$dy = pdx = pdx + xdp + \frac{\quad}{\sqrt[3]{(1+p^3)^2}}$$

ons tirons

$$0 = xdp + \frac{appdp}{\sqrt[3]{(1+p^3)^2}},$$

$$x = \frac{-app}{\sqrt[3]{(1+p^3)^2}} \quad \text{et} \quad y = \frac{a}{\sqrt[3]{(1+p^3)^2}}.$$

VI. Si l'on veut ici éliminer p, on n'a qu'à ajouter les cubes pour avoi

$$y^3 + x^3 = \frac{a^3\,(1-p^6)}{(1+p^3)^2} = \frac{a^3\,(1-p^3)}{1+p^3} = -a^3 + \frac{2a^3}{1+p^3}$$

te que

$$\frac{1}{1+p^3} = \frac{a^3+x^3+y^3}{2a^3},$$

tant

$$y = \frac{a}{\sqrt[3]{(1+p^3)^2}} = (a^3+x^3+y^3)^{\frac{2}{3}} : a\sqrt[3]{4}.$$

$$4a^3y^3 = (a^3+x^3+y^3)^2$$

$$0 = a^6 + 2a^3x^3 - 2a^3y^3 + x^6 + 2x^3y^3 + y^6$$

une ligne du sixième ordre. Mais outre celle-ci satisfait encore $dp = 0$
$= n$, à cause de la division faite par dp; et ce cas donne une infinité d
droites contenues dans cette équation

$$y = nx + a\sqrt[3]{(1+n^3)}.$$

VII. On voit que par la même méthode on résoudra aisément tous le
mes, qui conduiroient à de telles équations:

$$ydx - xdy = a\sqrt[n]{(\alpha dx^n + \beta dx^{n-\nu}dy^\nu + \gamma dx^{n-\mu}dy^\mu + \text{etc.})}$$

osant $dy = pdx$, on auroit

$$y = px + a\sqrt[n]{(\alpha + \beta p^\nu + \gamma p^\mu + \text{etc.})}$$

érentiant et divisant par dp,

$$y = \frac{n\alpha a + (n-\nu)\ \alpha\beta p^{\nu} + (n-\mu)\ \alpha\gamma p^{\mu} + \text{etc.}}{n\sqrt[n]{(\alpha + \beta p^{\nu} + \gamma p^{\mu} + \text{etc.})^{n-1}}}.$$

D'où, en éliminant p, on tirera une équation algébrique entre x et
qu'il y a aussi $dp = 0$ et $p =$ const. $= m$, les lignes droites ren
cette formule:

$$y = mx + a\sqrt[n]{(\alpha + \beta m^{\nu} + \gamma m^{\mu} + \text{etc.})}$$

satisferont également. Je passe donc à un autre problème.

PROBLEME II

*Sur l'axe AB trouver la courbe AMB (Fig. 2), telle, qu'ayant tir
quelconque M la tangente TMV, elle coupe en sorte les deux droite
tirées perpendiculairement sur l'axe AB, en deux points donnés A
rectangle formé par les lignes AT et BV soit partout de la même gra*

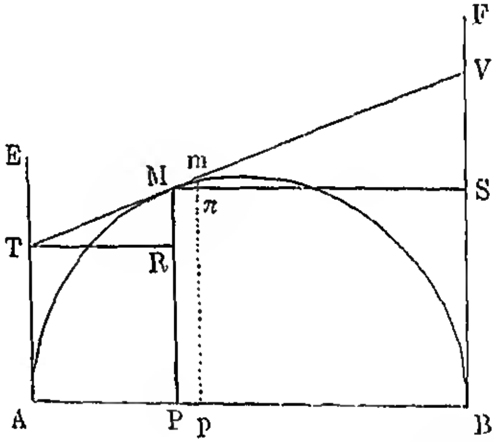

Fig. 2

XVIII. Soit l'intervalle donné $AB = 2a$, l'abscisse $AP =$
$PM = y$, et ayant tiré l'infiniment proche pm, on aura $Pp =$
$\pi m = dy$. Qu'on tire les droites TR et MS parallèles à l'axe AB,
blance des triangles $M\pi m$, TRM et MSV, à cause de

$$PB = MS = 2a - x,$$

fournira:

ous aurons:

$$AT = y - \frac{xdy}{dx} \text{ et } BV = y + \frac{(2a - x)\,dy}{dx},$$

e produit devant être constant $= cc$ fournira cette égalité:

$$\left(y - \frac{xdy}{dx}\right)\left(y - \frac{xdy}{dx} + \frac{2ady}{dx}\right) = cc.$$

IX. Si l'on vouloit traiter cette équation par la méthode ordinaire, on ntreroit bien des difficultés, et peut être n'arriveroit-on qu'après bien des s à l'équation intégrale. Mais, pour nous servir de l'autre méthode, s $dy = pdx$, pour avoir

$$(y - px)(y - px + 2ap) = cc$$

n:

$$yy + 2(a - x)\,py - 2appx + ppxx = cc \text{ ou}$$

$$yy + 2(a - x)\,py + (a - x)^2\,pp = cc + aapp,$$

'extraction de racine fournit:

$$y + (a - x)\,p = \sqrt{(cc + aapp)} \text{ ou}$$

$$y = -(a - x)\,p + \sqrt{(cc + aapp)}.$$

X. Différentions maintenant cette équation, au lieu d'en chercher rale, et nous obtiendrons:

$$dy = pdx = -(a - x)\;dp + \;pdx + \frac{aapdp}{\sqrt{(cc + aapp)}},$$

termes pdx se détruisant ensemble, la division par dp donnera:

$$a - x = \frac{aap}{\sqrt{(cc + aapp)}} \quad \text{ou} \quad x = a - \frac{aap}{\sqrt{(cc + aapp)}}$$

stituant cette valeur de $a - x$ dans celle de y, on aura

$$y = \frac{-aapp}{\sqrt{(cc + aapp)}} + \sqrt{(cc + aapp)} \quad \text{ou} \quad y = \frac{cc}{\sqrt{(cc + aapp)}}.$$

$$y = \frac{n\alpha a + (n-\nu)\ \alpha\beta p^\nu + (n-\mu)\ \alpha\gamma p^\mu + \text{etc.}}{n\sqrt[n]{(\alpha + \beta p^\nu + \gamma p^\mu + \text{etc.})^{n-1}}}.$$

D'où, en éliminant p, on tirera une équation algébrique entre x et y. qu'il y a aussi $dp = 0$ et $p = \text{const.} = m$, les lignes droites renferm cette formule:

$$y = mx + a\sqrt[n]{(\alpha + \beta m^\nu + \gamma m^\mu + \text{etc.})}$$

satisferont également. Je passe donc à un autre problème.

PROBLEME II

Sur l'axe AB trouver la courbe AMB (Fig. 2), telle, qu'ayant tiré d quelconque M la tangente TMV, elle coupe en sorte les deux droites tirées perpendiculairement sur l'axe AB, en deux points donnés A et rectangle formé par les lignes AT et BV soit partout de la même grand

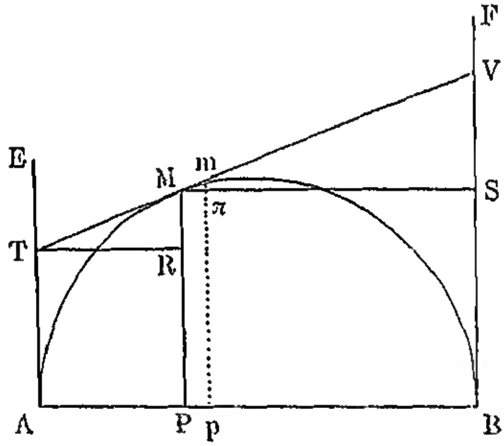

Fig. 2

XVIII. Soit l'intervalle donné $AB = 2a$, l'abscisse $AP = x$, $PM = y$, et ayant tiré l'infiniment proche pm, on aura $Pp = M$ $\pi m = dy$. Qu'on tire les droites TR et MS parallèles à l'axe AB, et blance des triangles $M\pi m$, TRM et MSV, à cause de

$$PB = MS = 2a - x,$$

fournira:

s aurons:

$$AT = y - \frac{xdy}{dx} \text{ et } BV = y + \frac{(2a - x)\,dy}{dx},$$

roduit devant être constant $= cc$ fournira cette égalité:

$$\left(y - \frac{xdy}{dx}\right)\left(y - \frac{xdy}{dx} + \frac{2ady}{dx}\right) = cc.$$

. Si l'on vouloit traiter cette équation par la méthode ordinaire, on roit bien des difficultés, et peut être n'arriveroit-on qu'après bien des l'équation intégrale. Mais, pour nous servir de l'autre méthode, $= pdx$, pour avoir

$$(y - px)(y - px + 2ap) = cc$$

$$yy + 2(a - x)py - 2appx + ppxx = cc \text{ ou}$$

$$yy + 2(a - x)py + (a - x)^2 pp = cc + aapp,$$

raction de racine fournit:

$$y + (a - x)p = \sqrt{(cc + aapp)} \text{ ou}$$

$$y = -(a - x)p + \sqrt{(cc + aapp)}.$$

Différentions maintenant cette équation, au lieu d'en chercher , et nous obtiendrons:

$$dy = pdx = -(a - x)\,dp + pdx + \frac{aapdp}{\sqrt{(cc + aapp)}},$$

mes pdx se détruisant ensemble, la division par dp donnera:

$$a - x = \frac{aap}{\sqrt{(cc + aapp)}} \quad \text{ou} \quad x = a - \frac{aap}{\sqrt{(cc + aapp)}}$$

ant cette valeur de $a - x$ dans celle de y, on aura

$$y = \frac{-aapp}{\sqrt{(cc + aapp)}} + \sqrt{(cc + aapp)} \quad \text{ou} \quad y = \frac{cc}{\sqrt{(cc + aapp)}}.$$

l'élimination de la quantité p se fera en ajoutant les quarré
formules, ce qui donnera:

$$\frac{(a-x)^2}{aa} + \frac{yy}{cc} = \frac{aapp+cc}{cc+aapp} = 1,$$

donc:

$$\frac{yy}{cc} = \frac{2ax-xx}{aa} \quad \text{ou} \quad y = \frac{c}{a}\sqrt{(2ax-xx)}.$$

D'où nous voyons que la courbe cherchée est une ellipse décrite et dont le demi-axe conjugué est $= c$, de sorte que dans une rectangle des tangentes AT et BV soit toujours égal au quarr conjugué.

XXII. Mais il est clair qu'outre cette ligne courbe il sati problème une infinité de lignes droites TV tellement tirées, q $AT \cdot BV$ soit $= cc$. Ces lignes droites se trouveront par le diviseu posé $= 0$, donne $p =$ const. $= n$. D'où nous aurons:

$$y = -n(a-x) + \sqrt{(cc+nnaa)}.$$

D'où, si $x = 0$, nous tirons

$$AT = -na + \sqrt{(cc+nnaa)},$$

et si $x = 2a$,

$$BV = na + \sqrt{(cc+nnaa)},$$

de sorte qu'on ait toujours

$$AT \cdot BV = cc,$$

quelque valeur que puisse avoir le nombre n.

PROBLEME III

Deux points étant donnés A et C (Fig. 3), trouver la ligne c que si l'on tire une tangente quelconque MV, qu'on y mene du pre perpendiculaire AV, et qu'on joigne de l'autre point C à V la droite CV soit partout de la même grandeur.

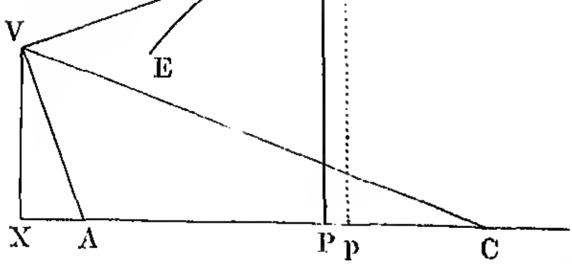

Fig. 3

XIII. Posons la distance donnée $AC = b$, et prenant cette ligne pour 'on y mene du point M l'appliquée MP, et son infiniment proche pm. $P = x$, et $PM = y$; et à cause de

$$Pp = M\pi = dx, \quad \text{et} \quad \pi m = dy,$$

$$Mm = \sqrt{(dx^2 + dy^2)} = ds.$$

sé, nous avons vû dans la solution du premier problème qu'on aura:

$$AV = \frac{y\,dx - x\,dy}{ds}.$$

s aussi du point V sur l'axe la perpendiculaire VX, et à cause des trian- ablables $Mm\pi$ et VAX nous aurons:

$$VX = \frac{dx\,(y\,dx - x\,dy)}{ds^2} \text{ et } AX = \frac{dy\,(y\,dx - x\,dy)}{ds^2}$$

ant:

$$CX = b + \frac{dy\,(y\,dx - x\,dy)}{ds^2}.$$

XIV. Soit maintenant la longueur donnée $CV = a$, et à cause de

$$CV^2 = CX^2 + XV^2$$

rons:

$$aa = bb + \frac{2b\,dy\,(y\,dx - x\,dy)}{ds^2} + \frac{(y\,dx - x\,dy)^2}{ds^2}$$

de $dx^2 + dy^2 = ds^2$;

lus:

$$\frac{(ydx - xdy)^2}{ds^2} + \frac{2bdy\,(ydx - xdy)}{ds^2} + \frac{bbdy^2}{ds^2} = aa - bb +$$

dont la racine quarrée est

$$\frac{ydx - xdy}{ds} + \frac{bdy}{ds} = \sqrt{\left(aa - \frac{bbdx^2}{ds^2}\right)}$$

ou bien en multipliant par ds

$$ydx - xdy + bdy = \sqrt{(aads^2 - bbdx}$$

XXV. Ici il est aussi évident, qu'on se plongeroit ennuyant, si l'on vouloit entreprendre la résolution de c méthode ordinaire. Je pose donc $dy = pdx$, et à cause de notre équation différentielle prendra cette forme

$$y - px + bp = \sqrt{\,(aa\,(1 + pp) - bb}$$

que je différentie encore, et posant pdx pour dy, j'aurai

$$pdx - pdx - xdp + bdp = \frac{aapdp}{\sqrt{(aa\,(1 + pp}}$$

qui étant divisée par dp donne:

$$b - x = \frac{aap}{\sqrt{(aa\,(1 + pp) - bb)}} \text{ ou } x = b - \frac{}{\sqrt{(aa\,(1}}$$

et

$$y = -(b - x)\,p + \sqrt{\,(aa\,(1 + pp) - bb)} = \frac{}{\sqrt{(aa\,(}}$$

XXVI. De là, pour éliminer p, je forme ces équation

$$\frac{b - x}{a} = \frac{ap}{\sqrt{(aa\,(1 + pp) - bb)}} \text{ et } \frac{y}{\sqrt{(aa - bb)}} = \frac{\sqrt{}}{\sqrt{(aa\,(}}$$

et ajontant les quarrés de ces formules, je trouve:

$$\frac{(b - x)^2}{aa} + \frac{yy}{aa - bb} = \frac{aa\,(1 + pp) - bb}{aa\,(1 + pp) - bb} = 1$$

qui est l'équation pour une ellipse, dont le centre est en A, et le demi grand axe $= CV$. Mais outre cette ellipse donne encore une infinité de lignes droites, comprises dans

$$y = -n\,(b - x) + \sqrt{\,(aa\,(1 + nn) - b}$$

ant tiré une tangente quelconque VMX, si l'on y mene des points A e rpendiculaires AV et BX, le rectangle de ces lignes AV·BX soit part même grandeur.

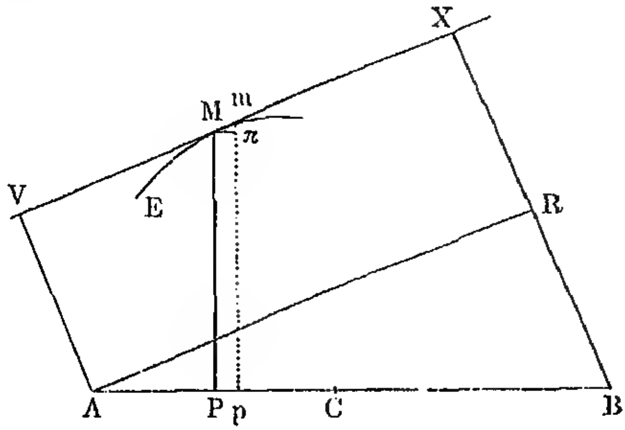

Fig. 4

XXVII. Soit la distance des points donnés $AB = 2b$, qu'on y tire ndiculaire MP, et l'infiniment proche mp; et qu'on nomme les co es: $AP = x$, $PM = y$, pour avoir

$$Pp = M\pi = dx, \pi m = dy \text{ et } Mm = \sqrt{(dx^2 + dy^2)} = ds.$$

posé, nous avons vû, qu'on aura

$$AV = \frac{ydx - xdy}{ds}.$$

n tire de plus AR, perpendiculaire sur BX, et la ressemblance des s $Mm\pi$ et ABR fournira

$$BR = \frac{2bdy}{ds},$$

y ajoutant

$$RX = AV = \frac{ydx - xdy}{ds}$$

aurons

$$BX = \frac{ydx + (2b - x)dy}{ds}.$$

$$\frac{(ydx - xdy)}{ds^2} + \frac{2bdy\,(ydx - xdy)}{ds^2} + \frac{\ }{ds^2} = aa - bb +$$

dont la racine quarrée est

$$\frac{ydx - xdy}{ds} + \frac{bdy}{ds} = \sqrt{\left(aa - \frac{bbdx^2}{ds^2}\right)}$$

ou bien en multipliant par ds

$$ydx - xdy + bdy = \sqrt{(aads^2 - bbdx}$$

XXV. Ici il est aussi évident, qu'on se plongeroit
ennuyant, si l'on vouloit entreprendre la résolution de c
méthode ordinaire. Je pose donc $dy = pdx$, et à cause d
notre équation différentielle prendra cette forme

$$y - px + bp = \sqrt{(aa\,(1 + pp) - bb}$$

que je différentie encore, et posant pdx pour dy, j'aurai

$$pdx - pdx - xdp + bdp = \frac{aapdp}{\sqrt{(aa\,(1 + p}}$$

qui étant divisée par dp donne:

$$b - x = \frac{aap}{\sqrt{(aa\,(1 + pp) - bb)}} \text{ ou } x = b - \frac{\ }{\sqrt{(aa\,(1}}$$

et

$$y = -(b - x)\,p + \sqrt{(aa\,(1 + pp) - bb)} = \frac{\ }{\sqrt{(aa}}$$

XXVI. De là, pour éliminer p, je forme ces équatio

$$\frac{b - x}{a} = \frac{ap}{\sqrt{(aa\,(1 + pp) - bb)}} \text{ et } \frac{y}{\sqrt{(aa - bb)}} = \frac{\sqrt{\ }}{\sqrt{(aa}}$$

et ajoutant les quarrés de ces formules, je trouve:

$$\frac{(b - x)^2}{aa} + \frac{yy}{aa - bb} = \frac{aa\,(1 + pp) - bb}{aa\,(1 + pp) - bb} =$$

qui est l'équation pour une ellipse, dont le centre est e
A, et le demi grand axe $= CV$. Mais outre cette ellips
donne encore une infinité de lignes droites, comprises dan

$$y = -n\,(b - x) + \sqrt{(aa\,(1 + nn) -}$$

s perpendiculaires AV et BX, le rectangle de ces lignes $AV \cdot BX$ soit … la même grandeur.

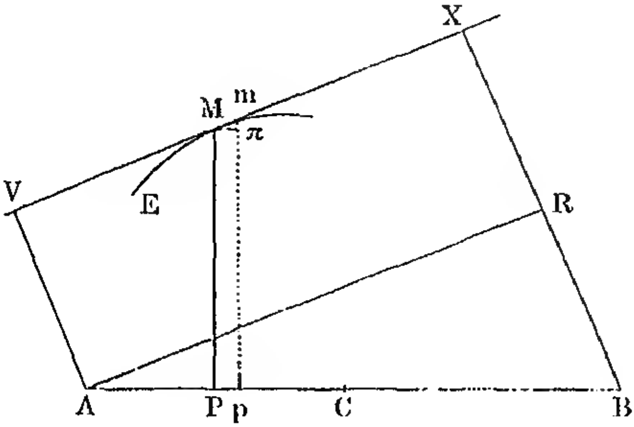

Fig. 4

XXVII. Soit la distance des points donnés $AB = 2b$, qu'on … erpendiculaire MP, et l'infiniment proche mp; et qu'on nomme l… onnées: $AP = x$, $PM = y$, pour avoir

$$Pp = M\pi = dx, \pi m = dy \text{ et } Mm = \sqrt{(dx^2 + dy^2)} = ds$$

ela posé, nous avons vû, qu'on aura

$$AV = \frac{y\,dx - x\,dy}{ds}.$$

Qu'on tire de plus AR, perpendiculaire sur BX, et la ressemblance … ngles $Mm\pi$ et ABR fournira

$$BR = \frac{2b\,dy}{ds},$$

… en y ajoutant

$$RX = AV = \frac{y\,dx - x\,dy}{ds}$$

ous aurons

$$BX = \frac{y\,dx + (2b - x)\,dy}{ds}.$$

XXVIII. Sans nous embarrasser de la méthode ordinair
$dy = pdx$, de sorte que

$$ds^2 = dx^2 (1 + pp),$$

et nous aurons:

$$(y - px)(y - px + 2bp) = cc(1 + pp)$$

qui se réduit à:

$$yy + 2(b - x) py - 2bppx + ppxx = cc(1 + p$$

ou à

$$yy + 2(b - x) py + (b - x)^2 pp = cc(1 + pp) +$$

dont la racine quarrée est

$$y + (b - x) p = \sqrt{(cc + (bb + cc) pp)}$$

et partant

$$y = -(b - x) p + \sqrt{(cc + (bb + cc) pp)}$$

XXIX. Différentions encore cette équation différentielle
$dy = pdx$ nous aurons:

$$pdx = -(b - x) dp + pdx + \frac{(bb + cc)\ pdp}{\sqrt{(cc + (bb + cc)}}$$

qui étant divisée par dp donne d'abord:

$$b - x = \frac{(bb + cc) p}{\sqrt{(cc + (bb + cc) pp)}}$$

ou bien

$$b - x = \frac{aap}{\sqrt{(cc + aapp)}},$$

posant pour abréger

$$bb + cc = aa.$$

De là nous tirerons:

$$y = -(b - x) p + \sqrt{(cc + aapp)} = \frac{cc}{\sqrt{(cc + aa}}$$

$a \qquad \sqrt{(cc + aapp)} \qquad c \qquad \sqrt{(cc + aapp)}$

s aurons en ajoutant les quarrés

$$\frac{(b-x)^2}{aa} + \frac{yy}{cc} = 1 .$$

XXX. Cette équation est, comme il est évident, pour une ellipse, don
ers sont dans les points A et B; et partant le centre au point du milie
lemi petit axe sera donc $= c$; et c'est au quarré duquel, que sera partout
ectangle $AV \cdot BX$: ce qui est aussi une propriété commune de l'ellipse. Or
si des lignes droites, qui satisfont au même problème, que le diviseur dp
s fournira, car posant $p = n$, l'équation pour toutes ces lignes droites

$$y = -n(b-x) + \sqrt{(cc + nnaa)}.$$

pourrois encore ajouter un grand nombre de problèmes semblables,
firmer ce paradoxe, mais ces quatre seront entierement suffisans pou
uver la vérité.

SECOND PARADOXE

XXXI. Le second paradoxe, que je m'en vai étaler, n'est pas moins
nant, puisqu'il est aussi contraire aux idées communes du calcul inté
s'imagine ordinairement, qu'ayant une équation différentielle quelcon
n'ait qu'à chercher son intégrale, et à lui rendre toute son étendue
itant une constante indéfinie, pour avoir tous les cas, qui sont compris
uation différentielle. Ou bien, lorsque cette équation différentielle e
ltat d'une solution d'un problème, on ne doute pas que l'équation intég
on en trouve par les règles ordinaires, ne renferme toutes les solu
sibles du problème: cela s'entend, lorsqu'on n'aura pas négligé l'add
constante, que toute intégration exige.

XXXII. Cependant il y a des cas, où l'intégration ordinaire nous co
ne équation finie, qui ne renferme pas tout ce qui étoit contenu dans l'é
différentielle proposée; quand même on ne néglige pas la constante
née. Cela doit paroitre d'autant plus paradoxe, plus on est accou

prescrites, n'épuise pas l'étendue de l'équation différentielle, le pr
mettra des solutions, que l'intégration ne fournira point, et partant
à une solution défectueuse, ce qui semble sans doute renverser le
ordinaires du calcul intégral.

XXXIII. Or il est fort aisé de proposer une infinité d'équatio
tielles, auxquelles répond un certain rapport entre les quantités var
est impossible de trouver par la voye d'intégration ordinaire. Soit, p
proposée cette équation différentielle:

$$xdx + ydy = dy\sqrt{(xx + yy - aa)},$$

et il est évident que l'équation finie

$$xx + yy - aa = 0$$

lui satisfait entierement. Car ayant de là $xdx + ydy = 0$, l'un
membre de l'équation différentielle évanouït de soi-même: ce e
marque indubitable, que cette équation finie

$$xx + yy = aa$$

est contenue dans l'équation différentielle proposée ou que le cercl
problèmes, qui conduisent à cette équation différentielle.

XXXIV. Cependant, quand nous intégrons cette équation di
nous ne trouverons nullement ce rapport $xx + yy = aa$; car, div
équation par $\sqrt{(xx + yy - aa)}$, que nous ayons:

$$\frac{xdx + ydy}{\sqrt{(xx + yy - aa)}} = dy\,,$$

l'intégrale est évidente, et même dans toute son étendue

$$\sqrt{(xx + yy - aa)} = y + c$$

ayant introduit la constante indéfinie c. Or il est clair que l'équation
vée $yy + xx = aa$ n'est pas absolument renfermée dans cette équ
grale, quelque valeur qu'on donne à la constante c.

$$xx - aa = 2cy + cc \text{ et } y = \frac{xx - aa - cc}{2c}$$

partant on croiroit qu'au problème proposé, qui aura conduit à cette équat satisfissent qu'une infinité de paraboles, contenues dans l'équation

$$y = \frac{xx - aa - cc}{2c},$$

on les différentes valeurs de c. Et puisqu'on a trouvé une infinité de p les, on doutera d'autant moins, qu'on ne soit arrivé à une solution compl pendant nous venons de voir qu'au même problème satisfait aussi le ce ntenu dans l'équation $xx + yy = aa$.

XXXVI. J'ai rencontré quelques autres cas de cette espèce dans aité du mouvement, où j'ai déjà remarqué ce même paradoxe, qu' uation différentielle renferme quelquefois des solutions, qui ne sont mprises dans l'équation intégrée[1]; j'y ai aussi donné une règle sûre, pa oyen de laquelle on peut trouver ces solutions contenues dans les équat fférentielles, qu'on ne sauroit plus tirer de l'équation intégrée. Cepend mme je n'y ai pas fait sentir assés évidemment l'importance de ce parad pourroit croire que c'est quelque bizarrerie dans des problèmes mécaniq i n'auroit plus lieu dans les problèmes de Géométrie; ou que ce ne se s un reproche, qu'on pourroit faire directement à l'Analyse même.

XXXVII. Pour l'exemple que je viens d'alléguer ici, comme il est fo fantaisie, on pourroit aussi douter, si ce cas se rencontre jamais dan lution d'un problème réel. Mais les mêmes exemples, que j'ai rapportés p laircir le premier paradoxe, serviront aussi à éclaircir celui-ci. Car le pren oblème demandant une courbe telle, que si l'on mene d'un point donné utes ses tangentes des lignes perpendiculaires, toutes ses perpendicula ient égales entr'elles; ce problème, dis-je, étant proposé, on voit d'ab 'un cercle décrit du point donné comme du centre avec un rayon égal oite, à laquelle toutes les perpendiculaires mentionnées doivent être éga tisfera au problème.

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1) Voir *Mechanica sive motus scientia* Tomus primus Caput V § 640, Petropoli 1736. *L* *RDI EULERI Opera omnia*, series *II*, vol. 1 p. 211. H.

où les variables x et y sont mêlées entr'elles, on a vû que par le m
substitution

$$y = u\sqrt{(aa - xx)}$$

elle se change en cette séparée,

$$\frac{du}{\sqrt{(uu-1)}} = \frac{a\,dx}{aa - xx},$$

dont l'intégrale prise dans toute son étendue étoit

$$u + \sqrt{(uu - 1)} = n\sqrt{\frac{a+x}{a-x}},$$

d'où j'ai tiré cette équation:

$$y = \frac{n}{2}(a + x) + \frac{1}{2n}(a - x)$$

laquelle ne renferme que des lignes droites, de sorte que le ce
cette heure entierement exclus de la solution du problème pr

XXXIX. Il en est de même du problème second, qui est r
nous avons vû par une ellipse exprimée par cette équation

$$y = \frac{c}{a}\sqrt{(2ax - xx)};$$

ce qui est aussi clair par les propriétés connues de l'ellipse. Or
cette équation différentielle:

$$\left(y - \frac{x\,dy}{dx}\right)\left(y - \frac{x\,dy}{dx} + \frac{2a\,dy}{dx}\right) = cc$$

nous en tirerons par l'extraction de racine:

$$\frac{dy}{dx} = \frac{(a-x)\,y + \sqrt{(aayy - cc\,(2ax - xx))}}{2ax - xx}$$

$$(2\,ax - xx)\,dy - (a - x)\,ydx = dx\sqrt{(aayy - cc\,(2\,ax -}$$

Or il est évident que l'équation

$$aayy - cc\,(2\,ax - xx) = 0$$

$$y = \frac{}{a} \sqrt{(2ax - xx)},$$

nt en différentiant leurs logarithmes:

$$\frac{dy}{y} = \frac{dx\,(a-x)}{2ax - xx}, \quad \text{ou} \quad (2ax - xx)\,dy - (a - x)\,y\,dx = 0,$$

que dans ce cas l'un et l'autre membre de l'équation différentielle
.

Mais, si nous traitons cette équation différentielle selon la méthode
e, et que nous posions

$$y = u\sqrt{(2ax - xx)},$$

oir

$$\sqrt{(aayy - cc\,(2ax - xx))} = \sqrt{(2ax - xx)\,(aauu - cc)}$$

$$dy = du\sqrt{(2ax - xx)} + \frac{u\,(a-x)\,dx}{\sqrt{(2ax - xx)}},$$

urs substituées changeront notre équation en cette forme:

$$x - xx)^{\frac{3}{2}} + u\,(a - x)\,dx\sqrt{(2ax - xx)} - u\,(a - x)\,dx\sqrt{(2ax - xx)}$$
$$= dx\sqrt{(2ax - xx)\,(aauu - cc)}$$

éduit maintenant à cette séparée,

$$\frac{du}{\sqrt{(aauu - cc)}} = \frac{dx}{2ax - xx} \quad \text{ou} \quad \frac{a\,du}{\sqrt{(aauu - cc)}} = \frac{a\,dx}{2ax - xx}$$

ntégrale prise généralement est

$$l\frac{au + \sqrt{(aauu - cc)}}{b} = \tfrac{1}{2}\,l\,\frac{x}{2a - x}$$

$$au + \sqrt{(aauu - cc)} = b\sqrt{\frac{x}{2a - x}} = \sqrt{\frac{b^2 x^2}{(2ax - xx)}}.$$

I. De là on trouvera aisément la valeur de u, qui sera:

$$au = \frac{cc\sqrt{(2ax - xx)}}{2bx} + \frac{bx}{2\sqrt{(2ax - xx)}}$$

on obtiendra

$$ay = \frac{cc\,(2ax - xx)}{2bx} + \frac{bx}{2} = \frac{acc}{b} + \frac{(bb - cc)\,x}{2b}$$

et il est évident que cette équation intégrale, quelque général à cause de la constante indéfinie b, ne renferme pas l'ellipse dé même accident aura aussi lieu dans les deux autres problèm lorsqu'on traitera les équations différentielles trouvées par la mét en cherchant son intégrale; où l'ellipse qui en fournit une bel sera plus comprise.

XLII. Mais voici la règle générale, par laquelle on peut ais ces cas de l'intégrale d'une équation différentielle proposée, qu l'intégration ordinaire. Soit z une fonction quelconque des d x et y, et Z une fonction quelconque de z. Soient de plus P, fonctions quelconques des variables x et y, et supposons qu'on à cette équation différentielle

$$Vdz = Z\,(Pdx + Qdy),$$

et il est clair, que la valeur $Z = 0$ satisfait à cette équation: car $z =$ const. et partant $dz = 0$, de sorte que dans le cas $Z = 0$ les do l'équation proposée évanouïssent.

XLIII. Par le moyen de cette règle on trouvera aisémen contient une solution du second problème; car étant parvenu à différentielle:

$$\frac{du}{\sqrt{(aauu - cc)}} = \frac{dx}{2ax - xx} \quad \text{ou} \quad du\,(2ax - xx) = dx\,\sqrt{(aa}$$

prenons u pour z, et la fonction $\sqrt{(aauu - cc)}$ pour Z, et l'équ sera remplie par l'égalité

$$Z = 0, \text{ ou } aauu - cc = 0,$$

d'où l'on tire $u = \frac{c}{a}$ et partant

$$y = \frac{c}{a}\sqrt{(2ax - xx)},$$

XLIV. Il est ici à remarquer, que ces mêmes cas inaccessibles à l'
tion ordinaire, sont *précisément* ceux, qu'une *différentiation réitérée*
ournis dans les éclaircissements du premier paradoxe. Et pour peu qu'
échisse, on s'apercevra que cet accord n'arrive pas par quelque hazar
pourra prononcer *en général, que toutes les fois qu'une équation diffé*
le, étant encore différentiée, conduit immédiatement à une équation f
te équation finie ne sauroit jamais être trouvée par la voye ordinair
tégration; mais que, pour la trouver, il faut appliquer la règle que je v
xposer. De là on voit donc que les deux paradoxes expliqués sont tellem
ensemble, que l'un renferme nécessairement l'autre.

XLV. La règle donc, suivant laquelle on juge ordinairement, si
uation différentielle est intégrée dans toute son étendue, ou non,
générale. On croit *communément, que lorsqu'on a intégré en sorte*
uation différentielle, que l'équation intégrale contient une constante i
e, qui ne se trouve pas dans la différentielle, alors l'équation intégrale
nplette, ou de la même étendue que la différentielle. *Mais nous voyons*
exemples rapportés que, quoique les équations trouvées par l'intégra
tiennent une telle constante, qui semble les rendre générales, les équat
férentielles renferment pourtant une *solution*, qui n'est pas comprise
tégrale[1]). Cette circonstance sur le critère des équations intégrales
ttes nous pourroit fournir un troisième paradoxe, s'il n'étoit pas dé
itement lié avec le précédent.

XLVI. Il peut donc souvent arriver, qu'il est même absolument im
e d'intégrer, ou même de séparer une équation différentielle proposé
t on peut néanmoins par la règle donnée trouver une équation finie
sfait à la question. Ainsi, si l'on étoit parvenu dans la solution d'un
me à une telle équation

$$aa\,(aa - xx)\,dy + aaxydx = (aa - xx)\,(ydx - xdy)\,\sqrt{(yy + xx - aa}$$

t on entreprendroit inutilement l'intégration, on seroit pourtant sûr
te équation finie

1) Voir *Institutiones calculi integralis* vol. I, § 546—576, 695—703; vol. II, § 821. LEON
ERI *Opera omnia*, series I, vol. 11 et 12. H

$$yy + xx - aa = 0,$$

tant l'un que l'autre membre de l'équation évanouït; ce qui devi
lorsqu'on met

$$y = z\sqrt{(aa - xx)},$$

car alors l'équation prendra cette forme:

$$aadz = (ydx - xdy)\sqrt{(zz - 1)},$$

et posant $Z = \sqrt{(zz - 1)}$ on aura par la règle donnée $\sqrt{(zz - 1)}$ -
et partant $yy + xx = aa$.

Commentatio 24[illegible] indicis Enestroemiani
...mentarii academiae scientiarum Petropolitanae 5 (1754/5), 1760, p. 84–144
Summarium ibidem p. 12–14

SUMMARIUM

...s Diophantea, ab auctore antiquo Graeco Diophanto[1]) sic dicta, potissimum ... numerorum refertur, atque huiusmodi quaestiones resolvere docet, quibus ...untur, qui certa ratione combinati evadant quadrati, vel cubi, aliusve ...i; veluti si quaerantur duo numeri, quorum quadrata addita iterum quadra...t, cuiusmodi numeri sunt 3 et 4, quorum quadrata 9 et 16 addita summam ...n quadratum. In genere igitur si hi numeri ponantur x et y, id requiritur, ...d quadratum, seu ut $\sqrt{(xx + yy)}$ sit numerus rationalis, atque semper in ...iusmodi problematum pervenitur ad tales formulas radicales, sive radix ... cubica, sive altioris gradus sit extrahenda, numerosque ibi signo implicatos ...i oportet, ut radix re vera extrahi possit, omnisque irrationalitas evanescat. ...dum Diophanteam ita definiri posse patet, ut sit methodus irrationalitatem ...l autem in Analysi communi sunt quantitates irrationales, id in Analysi ...t quantitates transcendentes, quae oriuntur, si qua formula differentialis ...re nequit, perinde atque ibi quantitates irrationales nascuntur, quando ex ...ula radicem extrahere non licet. Methodus igitur in Analysi infinitorum ...analoga in hoc versatur, ut quantitates formulam quandam differentialem ...a determinentur, ut integratio succedat, et integrale Algebraice exhiberi ...modi exempla statim occurrunt, quando vel curvae quadrabiles, vel rectifi...ntur, ubi positis coordinatis orthogonalibus x et y, eiusmodi relatio inter ...r, ut priori casu formula ydx, posteriori vero haec $\int\sqrt{(dx^2 + dy^2)}$ inte...ittat. Problema quidem, quo curvae quadrabiles quaeruntur, est facillimum, ...n ante inventam Analysin infinitorum solvi potuit, alterum vero de curvis

1) ...ntes vixit circa tertium saeculum p. Chr. n. H. D.

scilicet Diophanteae analoga; cuius principia Auctor in hac dissertatio
distincte proponit, sed etiam eo usque prosequitur, ut problemata, quae a
lyseos longe superare viderentur, nunc sine ullo fere labore resolvi queant.
haec methodus, quousque hic est exculta, plurimum adhuc a perfection
quiturque amplissimus campus, in quo Geometrae vires suas exerceant, atq
parte fines Analyseos proferant. Quanquam enim ab Auctore innumera
differentiales ad integrabilitatem sunt perductae, tamen plurimae supe
artificia hic tradita nondum sufficiunt; veluti si eiusmodi quaeratur relatio i
x et y, ut haec formula $\int\left(\frac{y\,dx}{x}+\frac{dy}{y}\right)$ integrationem admittat. Auctor fa
adhuc modo id praestare potuisse. Verum dantur sine dubio et in hac
omnem reductionem respuentes, quemadmodum etiam in methodo Diopha
formulae, quae nullo modo ad quadratum reduci possunt. Plurimum igit
stitisse censendus erit, qui, cuiusmodi formulae ad reducendum plane sint in
ostendere potuerit.

Quanta affinitas inter analysin finitorum et infinitorum int
utraque ex iisdem principiis sit nata, atque similibus operationibus
nemo ignorat, qui in utroque calculi genere vel leviter fuerit ve
latius autem hanc affinitatem patere deprehendi, quam vulgo p
quemadmodum in analysi finitorum ea methodus, quae Dioph
refertur, insignem occupat locum, ita etiam in analysi infinitor
dari calculi genus observavi, qui methodo Diophanteae penitu
similibusque operationibus absolvatur. Quanquam autem huius
analysi infinitorum nonnulla iam passim occurrunt specimina, quo
mentionem sum facturus, tamen in iis nulla certa solutionis via
solutiones casu potius ac divinatione inventae videntur, ita ut i
certa ac tuta methodus adhuc desideretur. Quamobrem mihi q
calculi genus in medium proferre videor, qui omnino dignus sit, i
excolendo Geometrae vires suas exerceant. Mihi quidem tantum c
eius fundamenta eruere, quae autem iam ad plurima satis illustria
recondita problemata solvenda sufficiunt; eaque hic quantum po
et dilucide exponam, quo aliorum, qui in hoc genere elaborare vo
promoveatur ac sublevetur.

Ut igitur primum indolem et naturam huius novae methodi

1) Vide notas, p. 76.

que ex infinita solutionum multitudine eas elicere docet, quae quantitat
ionalibus contineantur, ita nova nostra methodus quoque nonnisi inde
nata problemata complectitur, et cum discrimini, quod in analysi finito
er quantitates rationales et surdas statui solet, in analysi infinitorum
rimen inter quantitates algebraicas ac transcendentes respondeat, no
strae methodi vis in hoc erit posita, ut ex infinita cuiusque problem
utionum copia eae secernantur, quae quantitatibus algebraicis contine
r. Huiusmodi igitur problemata indeterminata methodo nostrae sunt prop
orum solutio in genere concepta formulas transcendentes, seu integ
volvit, ex quibus deinceps eos casus elici oportet, quibus quantitates
anscendentes in algebraicas abeunt, seu, quod eodem redit, formulae
egrales integrationem admittant.

Per exemplum tam natura huius novae methodi, quam eius affinitas
ethodo Diophantea clarius elucescet. Uti enim in methodo Diophantea qu
et, quomodo quantitates x et y inter se debeant esse comparatae, ut h
mula $\sqrt{(xx + yy)}$ fiat rationalis, ita in nova nostra methodo huic sin
t ista quaestio, qua inter quantitates variabiles x et y ea quaeritur cond
formula specie transcendens $\int\sqrt{(dx^2 + dy^2)}$ fiat algebraica, seu ut h
mulae valor algebraice exhiberi queat. Manifestum est, hoc problem
od instar exempli attulimus, quaeri curvas algebraicas, quae sint re
abiles; relatio enim inter x et y, quae coordinatas curvae denotabunt, re
ur algebraica, unde quaestio circa curvas algebraicas versatur, et cum h
rvae arcus indefinite per $\int\sqrt{(dx^2 + dy^2)}$ exprimatur, quoties ista form
gebraica reddetur, toties ipsa curva erit rectificabilis.

Simili modo si omnes eae curvae algebraicae desiderentur, quae sint quad
es, perspicuum est, quaestionem huc redire, ut eae relationes inter qua
tes variabiles x et y assignentur, quibus haec formula integralis $\int y dx$ i
ationem admittat, atque ad valorem algebraicum perducatur.

Etsi autem hic potissimum quantitates algebraicae sunt proposi
rindo atque in methodo Diophantea quantitates rationales spectari sol
men eo quoque referendae sunt eiusmodi quaestiones, quibus form
acpiam integrales non algebraice exprimi, sed propositam quandam tr
ndentium quantitatum speciem implicare debent; veluti si quaerantur e
odi curvae algebraicae, quarum rectificatio non algebraice perfici queat,
quadratura circuli pendeat. Variae enim transcendentium quantita

venire docet, quaque ad eas curvas, quarum rectificatio a
pendeat, inveniendas aptam fore, id quod ex sequentibus el

Huiusmodi problema iam ante complures annos a Celeb.
propositum[1]), quo eiusmodi curvam algebraicam quaesivera
rectificabilis, sed cuius rectificatio a quadratura datae curv
tamen nihilo minus tot, quot lubuerit, arcus absolute rect
Propositione huius problematis tum temporis summus A
IOH. BERNOULLIUS b. m. adeo obstupuit, ut non solum
HERMANNO solutum esse non crediderit, sed etiam sagac
longe superare pronunciaverit; quod quidem nemini mirum
illo tempore nulla plane ullius methodi vestigia patuissent, c
problemata tractari possent. HERMANNUS etiam eius solut
ambages ex quadam linearum curvarum contemplatione ha
intuitu nihil plane emolumenti ad propositum expectare lic
nato ad solutionem ante pervenisset, quam de ipso prob
Visa autem ista HERMANNI solutione, BERNOULLIUS etiam
solutionem ex sola analysi petitam: sed cuius fundament
absconditum, ut divinatione potius, quam ulla certa via, f
tionem continentes eruisse videatur.

Cum hoc problema non solum ob summam, qua imp
tatem, sed etiam ob eximium usum, qui inde in analysin red
omnium tum temporis Geometrarum admirationem excita
quantum constat, in certam atque ad huiusmodi problema
methodum inquisivit, qua novus omnino analyseos infinito
aperiretur. Ego igitur longo post intervallo fortasse primus
huius methodi cogitare coepi, quorum beneficio memorati
solutio directe sine ambagibus ac divinatione obtineri poss
regulas quasdam non contemnendas, quae ad novae istius m
iacienda idonea sunt visa, earumque ope non solum plures
quod erat agitatum, solutiones sum adeptus, sed etiam non
generis problemata dedi soluta, cuiusmodi est illud, cuius s
in Dissertatione de duabus curvis algebraicis[3]) ad commu

1) Vide notam 1, p. 76.
2) Vide notam 2, p. 76.
3) Vide L. EULERI Commentationem 48 huius voluminis, p. 76.

a celavi, cum mihi esset propositum prima quasi huius methodi elem... sim explicare, quo eorum usus amplissimus clarius perspiciatur, n... ad hoc unicum problema adstricta videantur. Fateri quidem statim co... levem adhuc partem tantum huius novae methodi, quam hic prop... eleasse; verum his principiis stabilitis, non dubito, quin ea mox ma... ementa sit acceptura.

Divisio huius methodi in partes secundum naturam formularum integral... rum valores algebraici sunt efficiendi, commodissime instituetur. Cum e... per relatio inter duas quantitates variabiles x et y quaeratur, ut una plu... mulae integrales, quae has variabiles una cum suis differentialibus involv... braicos obtineant valores, huiusmodi formulas in sequentes ordines di... convenict:

Ordo primus continebit huiusmodi formulas $\int Zdx$, ubi Z est functio q... que algebraica ambarum quantitatum x et y.

Ad ordinem secundum refero eas formulas $\int Zdx$, in quibus posito $dy =$... ra Z est functio non solum ipsarum x et y, sed etiam ipsius p. Ubi no... n est, non solum formulam $\int Zdx$, sed etiam hanc $\int pdx = y$ algebr... ore debere valores. Huc reducuntur eae formulae integrales, in quibus a... rentialia dx et dy occurrunt, veluti $\int \sqrt{(dx^2 + dy^2)}$, quae posito $dy =$... hanc formam $\int dx \sqrt{(1 + pp)}$ revocatur.

Ordo porro tertius eiusmodi comprehendet formulas integrales, in q... m differentialia secundi gradus insunt, quae autem, ponendo $dy = p$... $= qdx$, ad hanc formam $\int Zdx$ perducentur, ubi littera Z erit functio q... tum x, y, p et q. His igitur casibus non solum formulae $\int Zdx$, sed e... um formularum $\int pdx$ et $\int qdx$ valores algebraici effici debebunt.

Ordo quartus complectetur eas formulas integrales, quae quantita... t y differentialia etiam tertii gradus involvunt; hucque ad formam $\int$... ucentur, ponendo $dy = pdx$, $dp = qdx$ et $dq = rdx$, ubi quantitas Z... ebit praeter quantitates x et y etiam has p, q et r. Hincque simul ... uentium ordinum intelligitur.

Praeter hos ordines peculiarem classem constituunt eiusmodi form... dx, in quibus Z non solum quantitates algebraicas x, y, p, q etc. uti i... inibus, continet, sed etiam formulas integrales complectitur, veluti si ...

$$\int x\,dx \int dx\sqrt{(1+pp)}$$

efficienda sit algebraica, pro quo relatio inter quantitates x et p def
In hoc exemplo primum patet, cum sit $dy = pdx$, valorem hui
$\int pdx$ esse debere algebraicum. Deinde etiam valorem huius

$$\int dx\sqrt{(1+pp)}$$

esse oportebit algebraicum, qui si ponatur $= s$, tandem haec for
ad valorem algebraicum erit perducenda, ita ut unica haec formul

$$\int xdx \int \sqrt{(dx^2+dy^2)}$$

reductionem harum trium formularum

$$\text{I. } \int pdx = y; \quad \text{II. } \int dx\sqrt{(1+pp)} = s; \quad \text{III. } \int xsdx$$

ad valores algebraicos requirat. Ex quo intelligitur, etiam huiusm
ad ordines ante enumeratos revocari posse.

Totum igitur negotium novae huius methodi, quam examini A
propono, in hoc consistit, ut eiusmodi relatio inter binas variabile
vestigetur, quae unam pluresve formulas integrales, cuiusmodi
supra descriptis sum complexus, algebraicas reddat[1]). Hic autem
problemata occurrunt difficillima, a quorum solutione equidem
sum remotus, sed etiam fortasse eiusmodi excogitari possunt, q
plane solutionem admittunt; omnino uti usu venire solet in pr
ad methodum Diophanteam pertinentibus. Unde etiam sine dubio
tudo locum inveniet, ut alia problemata solutionem generalem, al
tum solutiones speciales permittant.

Huiusmodi igitur problemata hic tantum proferam, quorum
inveni, ut hoc modo specimen ac prima quasi elementa novae me
ulterius excolendam propono, exhibeam, quae etsi exiguam tan
huius methodi constituere videntur, tamen viam, qua ulterius pro
patefacient. Certa autem inde earum operationum ratio perspic
directe nihilque divinationi tribuendo ad solutiones eorum problem
ante commemoravi, perducant.

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1) Vide notam p. 31.

quadratura pendebit integratio alterius formulae $\int ydx$, ab eadem q… rius $\int xdy$ integratio pendebit.

Demonstratio est manifesta, cum sit

$$\int ydx = xy - \int xdy,$$

le patet, si formula $\int xdy$ fuerit vel algebraica, vel datam quadratura… eans, eandem quoque naturam habere alteram formulam $\int ydx$.

COROLLARIUM

2. Simili modo integratio huius formulae $\int yxdx$, vel huius $\int yx^n dx$ p… ab integratione huius $\int xxdy$, vel huius $\int x^{n+1}dy$, ob

$$\int yxdx = \tfrac{1}{2}yxx - \tfrac{1}{2}\int xxdy,$$

ob

$$\int yx^n dx = \frac{1}{n+1}yx^{n+1} - \frac{1}{n+1}\int x^{n+1}dy,$$

le perspicitur hoc lemma latissime patere, eiusque ope formulas c… eris, quae integrabiles sint reddendae, in alias transformari posse.

SCHOLION

3. Lemma hoc, quantumvis leve ac triviale videatur, tamen praecip… tinet fundamentum novae illius methodi, quam sum adumbraturu… m proposita formula integrali quaecunque $\int YdX$ alia detur $\int VdZ$, …

$$A\int YdX + B\int VdZ$$

antitas algebraica, manifestum est, harum duarum formularum $\int Y$… dZ rationem ita esse comparatam, ut si altera fuerit integrabilis, … eram fore integrabilem, et a quanam quadratura alterius integratio pen… eadem quadratura etiam alterius integrationem pendere. Resolutio a… ecipuorum problematum ad hanc methodum pertinentium absol… nea formularum integralium, ad quas pervenitur, transformatione.

PROBLEMA 1

4. Invenire omnes curvas algebraicas, quae sint quadrabiles; seu cam… riabiles x et y relationem in genere definire, ut formula $\int ydx$ fiat integra…

curvae area $= \int y dx$, cuius valorem algebraicum esse op
facillime impetratur. Denotet enim X functionem quam
ipsius x, huicque functioni X aequalis ponatur area $\int yd$

$$\int y dx = X,$$

erit, differentialibus sumendis,

$$y dx = dX, \text{ unde fit } y = \frac{dX}{dx};$$

sicque applicata y aequabitur functioni algebraicae ipsius
algebraica, eiusque area $\int y dx$, cum sit $= X$, algebraice

ALITER

Cum sit area

$$\int y dx = yx - \int x dy,$$

ponatur $\int x dy$ functioni cuicunque ipsius y, quae sit $=$

$$\int x dy = Y, \text{ unde fit } x = \frac{dY}{dy},$$

ita ut iam abscissa x functioni algebraicae ipsius y aeq
algebraica. Posita autem $x = \frac{dY}{dy}$, erit curvae area

$$\int y dx = yx - Y = \frac{y dY}{dy} - Y,$$

ideoque etiam algebraica.

COROLLARIUM 1

5. Si X in priori solutione, vel Y in posteriori, non fue
ipsius x, vel y, sed transcendens, ita tamen ut $\frac{dX}{dx}$, vel
braica, curva quidem erit algebraica, sed eius quadratu
cendente exprimetur.

COROLLARIUM 2

6. Scilicet si in priori solutione sit

$$X = P + \int Q dx,$$

$$y = \frac{dP}{dx} + Q$$

quidem algebraica, sed eius area

$$\int y dx = P + \int Q dx$$

antitate transcendente $\int Q dx$ pendebit.

COROLLARIUM 3

7. Simili modo in altera solutione si ponatur

$$Y = P + \int Q dy,$$

entibus P et Q functionibus algebraicis ipsius y, ita tamen ut $\int Q dy$ si
itas transcendens, aequatio pro curva

$$x = \frac{dP}{dy} + Q$$

algebraica, sed area, quae erit

$$\int y dx = \frac{y dP}{dy} + yQ - P - \int Q dy$$

antitate transcendente $\int Q dy$ pendebit.

SCHOLION

8. Uti huius problematis solutio est facillima nulloque artificio indiget
ns problema, quod quidem alius est naturae, adiungam, cuius ver
o in aliis problematibus, quae ad hanc methodum referri solent, insignem
praestabit. Veluti si quaerantur curvae algebraicae generatim non
cabiles, quae tamen, quot lubuerit, habeant arcus rectificabiles; aliaev
generis quaestiones proponantur, principium solutionis ex sequent
omate erit petendum.

PROBLEMA 2

9. Invenire curvas algebraicas in genere non quadrabiles, sed quarum
atura generalis datam quantitatem transcendentem involvat, in quibu
n, quot lubuerit, areas absoluto quadrabiles assignare liceat.

redire, ut eiusmodi formula transcendens $\int Qdx$ investigetur casibus, veluti si ponatur $x = a$, $x = b$, $x = c$ etc., evanesc quantitas

$$X = P + \int Qdx,$$

quae in genere est transcendens, quippe formulam $\int Q$ algebraica, nempe $= P$. Hoc ut efficiatur, statuatur

$$\int Qdx = \int udx - \int vdz,$$

ubi v talis sit functio ipsius z, qualis u est ipsius x, ita $\int vdz$ similem quantitatem transcendentem exhibeant, debet. Sit autem z eiusmodi functio ipsius x, ita ut casib $x = b$, $x = c$ etc., quot lubuerit, fiat $z = x$, ideoque et $v =$ est, his iisdem casibus fore $\int vdz = \int udx$, hincque $\int Qdx$ formetur ista functio ipsius x

$$x^n - (a + b + c + \text{etc.})x^{n-1} + (ab + ac + bc + \text{etc.})x^{n-2} - (a$$

quae brevitatis gratia vocetur $= S$, ita ut aequatio $S =$ $x = a$, $x = b$, $x = c$ etc. eos scilicet ipsos valores absci absolute quadrabilis respondere debet. Tum vero statuatu

$$z - x = S,$$

atque manifestum est, iisdem casibus $x = a$, $x = b$, x omnino uti requiri ad nostrum propositum ostendimus. generalius satisfiet, si ponamus

$$z - x = ST,$$

dummodo $ST = 0$ alias non praebeat radices reales, nisi scilicet $x = a$, $x = b$, $x = c$ etc. Hanc ob rem si S den tionem ipsius x, ut aequatio $S = 0$ alias non habeat rad sunt propositae, scilicet $x = a$, $x = b$, $x = c$ etc., quod s fieri potest, tum sumatur

$$z - x = S, \text{ seu } z = x + S.$$

Quo facto, si $\int udx$ eam quantitatem transcendentem ex quadratura in genere pendere debet, pro v substituatur

$$Q = u - \frac{\quad}{dx}.$$

am enim si construatur curva algebraica, cuius abscissae $= x$ respond
plicata

$$y = \frac{dP}{dx} + u - \frac{vdz}{dx},$$

us area in genere erit

$$\int ydx = P + \int udx - \int vdz,$$

ndebit scilicet a quantitate transcendente $\int udx$, cui altera $\int vdz$ est sim
hilo vero minus casibus $x = a$, $x = b$, $x = c$ etc. eius area algebraice
imetur, fietque $\int ydx = P$. Hoc ergo modo effici potest, ut curva praecise
iot quis voluerit, obtineat areas quadrabiles, neque plures, neque paucio

COROLLARIUM 1

10. Cum v talis sit functio ipsius z, qualis u est ipsius x, ita ut v obtine
u, si loco x scribatur z, sequitur etiam v talem esse functionem ipsiu
ualis z est ipsius x. Quare cum sit $z = x + S$, sequitur v obtineri ex u, si
scribatur $x + S$.

COROLLARIUM 2

11. Quoniam igitur quantitas v resultat ex functione u, si loco x scrib
$+ S$, ex proprietate functionum alias demonstrata sequitur fore

$$v = u + \frac{Sdu}{dx} + \frac{S^2ddu}{1\cdot 2dx^2} + \frac{S^3d^3u}{1\cdot 2\cdot 3dx^3} + \frac{S^4d^4u}{1\cdot 2\cdot 3\cdot 4dx^4} + \text{etc.}$$

osito elemento dx constante, sed cum haec expressio in infinitum sit c
uanda, praestat valorem ipsius v actuali substitutione definire.

EXEMPLUM

12. *Invenire curvam algebraicam, cuius quadratura indefinita pend
uadratura circuli, cuius vero area abscissae $x = a$ respondens algebraic
ibeatur.*

Ut quadratura curvae indefinita a quadratura circuli pendeat, pon

$$u = \sqrt{(2fx - xx)},$$

$$z = x + na - nx = na - (n-1)\,x.$$

Ergo ob

$$v = \sqrt{(2fz - zz)}$$

erit

$$v = \sqrt{(2naf - 2(n-1)fx - nnaa + 2n(n-1)ax - (n$$

Ponatur, ut haec formula simplicior evadat, $2f = na$, eritque

$$v = \sqrt{(n(n-1)ax - (n-1)^2 xx)},$$

et ob $dz = -(n-1)\,dx$ habebitur

$$Q = \sqrt{(nax - xx)} + (n-1)\sqrt{(n(n-1)ax - (n-1$$

ac pro curva erit

$$y = \frac{dP}{dx} + \sqrt{(nax - xx)} + (n-1)\sqrt{(n(n-1)ax - (n-$$

area vero erit

$$\int y\,dx = P + \int dx\sqrt{(nax - xx)} + (n-1)\int dx\sqrt{(n(n-1)ax-$$

Verum hic notandum est, quemadmodum integrale $\int u\,dx$ ita ca evanescat posito $x = 0$, ita quoque integrale $\int v\,dz$ ita capi debere, posito $z = 0$. Quamobrem ut tota area evanescat posito $x = 0$, quoque fiat $z = 0$ hoc casu; alioquin enim expressio areae $\int y\,dx$ c quantitatem constantem portionem areae circularis denotante $x = a$ destrueretur. Huic autem incommodo occurretur, si pr sumatur functio, quae posito $x = 0$ evanescat. Sit ergo

$$S = \frac{nx}{a}(a - x),$$

et

$$z = x + \frac{nx}{a}(a - x), \text{ et } v = \sqrt{(2fz - zz)},$$

atque quaesito satisfiet modo solito. Ponatur, ut expressio fiat $n = -1$, ut sit

$$z = \frac{xx}{a} \quad \text{et} \quad v = \sqrt{\left(\frac{2fxx}{a} - \frac{x^4}{aa}\right)} = \frac{x}{a}\sqrt{(2af - xx)},$$

$dz = \frac{2\,xdx}{a}$, atque area fiet

$$\int ydx = P + \int dx\sqrt{(2fx - xx)} - 2\int \frac{xxdx}{aa}\sqrt{(2af - xx)},$$

c, qualiscunque P fuerit functio ipsius x, in genere semper a quadra uli pendebit, casu autem $x = a$ area fiet algebraica $= P$.

SCHOLION

13. Circumstantia haec ratione constantis ad areae expressionem adi , ne ea ipsa sit trancendens, in omnibus exemplis probe est observa nc in finem functio S non solum ita accipi debebit, ut casibus prop $= a$, $x = b$, $x = c$ etc. evanescat, sed etiam casu $x = 0$ evanescere deb od quidem per se est perspicuum: nam quia omnis curvae aream absc nescenti $x = 0$ respondentem nihilo aequalem assumimus, ideoq nscendentibus quantitatibus vacuam, evidens est, quotcunque casus p sint, quibus area fiat algebraica, iis semper superaddendum esse c $= 0$, sicque functio S ita comparata esse debebit, ut non solum casibus x $= b$, $x = c$ etc., qui sunt propositi, sed etiam casu $x = 0$ fiat $S = 0$.

PROBLEMA 3

14. Si Z sit functio quaecunque algebraica binarum variabilium x nire relationem algebraicam inter x et y, ut formula integralis $\int Zdx$ icum obtineat valorem.

SOLUTIO

Etsi problema hoc multo latius patere videtur, quam primum, tame ntio non est difficilior. Ponatur enim $\int Zdx$ functioni cuicunque algeb us x, quae sit $= X$, aequale, eritque

$$Zdx = dX \quad \text{et} \quad Z = \frac{dX}{dx},$$

cum $\frac{dX}{dx}$ sit quoque functio algebraica ipsius x, habebitur aequatio ica inter x et y, qua earum relatio algebraice definietur: indeque er othesin $\int Zdx = X$.

$$X = P + \int Qdx, \text{ ita ut } \frac{dX}{dx} = \frac{dP}{dx} + Q$$

sit nihilominus functio algebraica ipsius x; tum orietur
aequatione

$$Z = \frac{dP}{dx} + Q$$

expressa, sed valor integralis inde oriundus $\int Zdx$ non
functionem transcendentem $\int Qdx$ involvet.

COROLLARIUM 2

16. Si pro Q eiusmodi quantitatem substituamus, q
praecedente descripsimus, tum valor quidem indefinitus f
algebraicus, sed a quadratura quapiam data pendebit. Ho
effici potest, ut eius valor tot casibus, quot lubuerit, e
$x = a$, $x = b$, $x = c$ etc. fiat algebraicus. Ubi quidem no
his casibus superaddendum esse semper casum $x = 0$.

SCHOLION

17. Si igitur unica proponatur formula integralis ad
reducenda, eaque pertineat ad ordinem primum, tum
difficultate. Atque simul pari opera effici potest, ut illi
a data quadratura pendeat, atque insuper ut tot, quot
algebraicum obtineat valorem. Antequam igitur ad form
num progrediar, eiusmodi problemata proponam, quibus
lae ordinis primi simul ad valores algebraicos sint reduce
bus V et Z functionibus ipsarum x et y, valores harum
$\int Vdx$ et $\int Zdx$ vel plurium huiusmodi algebraici sint effici
omnia animadverto, haec problemata in genere concep
solubilia videri, sed nonnisi sub certis conditionibus, qu
sint praeditae, solutionem admittere. Quibus igitur ca
solutionem pervenire licuerit, hic exponam.

eam inter variabiles x et y, ut ambae hae formulae $\int yPdx$ et $\int yQdx$ es algebraicos adipiscantur.

SOLUTIO

Ponatur utraque formula seorsim aequalis quantitati cuicumque algeae, scilicet

$$\int yPdx = L \quad \text{et} \quad \int yQdx = M.$$

ergo fiet

$$y = \frac{dL}{Pdx} \quad \text{et} \quad y = \frac{dM}{Qdx}$$

ue

$$\frac{P}{Q} = \frac{dL}{dM},$$

nt L et M functiones novae cuiuspiam variabilis z, ita ut $\frac{dL}{dM}$ sit functio raica huius variabilis z. Ope aequationis ergo inventae

$$\frac{P}{Q} = \frac{dL}{dM}$$

ipsius x, cuius functio est $\frac{P}{Q}$, per z expressus reperietur, ita ut inde prom sit x aequale functioni cuipiam ipsius z. Qua inventa obtinebitur ue valor ipsius y per functionem quampiam ipsius z expressus, ope formulae

$$y = \frac{dL}{Pdx} \quad \text{vel} \quad y = \frac{dM}{Qdx},$$

e utraque variabilis x et y per novam variabilem z determinabitur, idque raice; unde relatio inter x et y quaesita innotescet. Ex his autem valoribus uti assumsimus,

$$\int yPdx = L \quad \text{et} \quad \int yQdx = M,$$

ue scilicet functioni algebraicae ipsius z aequalis.

ALIA SOLUTIO

Ponatur ut ante altera formula $\int yPdx$ quantitati cuipiam algebraicae qualis, seu

$$\int yQdx = \int \frac{Q}{P} dL,$$

quae algebraica reddenda restat. Iam vero per lemma pra

$$\int \frac{Q}{P} dL = \frac{LQ}{P} - \int Ld \cdot \frac{Q}{P}.$$

Sicque formula $\int Ld \cdot \frac{Q}{P}$ ad algebraicum valorem reduci d $d \cdot \frac{Q}{P}$ huiusmodi formam Xdx esse habiturum, ubi sit X fu Ponatur ergo $\int Ld \cdot \frac{Q}{P}$ functioni cuicunque ipsius x, quae

$$L = \frac{dV}{d(Q:P)}$$

functioni scilicet ipsius x. Invento autem valore ipsius L e

$$\int yPdx = L; \quad \int yQdx = \frac{LQ}{P} - V$$

atque variabilis y ita definietur per x, ut sit $y = \frac{dL}{Pdx}$, ex

$$L = dV : d \cdot \frac{Q}{P};$$

hoc ergo modo immediate, nulla alia nova variabili i variabilem y per x dedimus determinatam.

COROLLARIUM 1

19. Cum in priori solutione altera variabilis x definiri

$$\frac{P}{Q} = \frac{dL}{dM},$$

altera vero sit

$$y = \frac{dL}{Pdx},$$

sicque utraque per novam variabilem z, cuius L et M sunt f

$$\int yPdx = L \quad \text{et} \quad \int yQdx = M.$$

COROLLARIUM 2

20. Per eandem ergo solutionem sumendis pro L et M functionibus trans
dentibus ipsius z, ita tamen ut

$$\frac{dL}{dz} \text{ et } \frac{dM}{dz}$$

functiones algebraicae, effici poterit, ut integratio utriusque formula
positae

$$\int yPdx \quad \text{et} \quad \int yQdx$$

ta quadratura pendeat; vel ut altera sit algebraica, altera vero datar
draturam involvat.

COROLLARIUM 3

21. Si ambae hae formulae debeant esse algebraicae, solutio posterio
dem praestat usum; sumta enim pro V functione quacunque algebraic
us x, erit

$$L = dV : d \cdot \frac{Q}{P}$$

quo functio algebraica ipsius x; tum vero si statuatur altera variabili
$\frac{dL}{Pdx}$, erit

$$\int yPdx = L \quad \text{et} \quad \int yQdx = \frac{LQ}{P} - V$$

$$\int yPdx = \frac{dV}{d \cdot \frac{Q}{P}} \quad \text{et} \quad \int yQdx = \frac{QdV}{Pd \cdot \frac{Q}{P}} - V.$$

COROLLARIUM 4

22. Sin autem in hac solutione pro V capiatur functio transcendens ipsiu
a tamen ut $\frac{dV}{dx}$ sit functio algebraica, ob $\frac{d(Q:P)}{dx}$ etiam functionem alge
eam fiet quoque

$$L = dV : d \cdot \frac{Q}{P},$$

valor fiet algebraicus, atque altera tantum $\int yQdx$ a praesc

pendebit.

COROLLARIUM 5

23. Per hanc igitur alteram solutionem effici non p
formula integralis proposita datam quadraturam involvat,
semper reperitur algebraicus. Quare si utraque debeat lud
cendentem, solutione priore erit utendum.

EXEMPLUM

24. *Invenire curvas algebraicas, in quibus non solum a*
areae momentum $\int yxdx$ algebraice exhiberi possit.

Per priorem solutionem ponatur:

$$\int ydx = L \quad \text{et} \quad \int yxdx = M$$

erit

$$y = \frac{dL}{dx} = \frac{dM}{xdx},$$

unde fit

$$x = \frac{dM}{dL} \quad \text{et} \quad y = dL : d\left(\frac{dM}{dL}\right),$$

ubi pro L et M functiones quaecumque algebraicae novae
possunt. Nihil ergo impedit, quo minus statuatur $L = z$ e
functio quaecumque ipsius z, quae sit $= Z$, quo facto erit

$$x = \frac{dZ}{dz}$$

et sumto elemento dz constante

$$y = \frac{dz^2}{ddZ}.$$

Per alteram solutionem ponatur

$$\int ydx = L,$$

ut sit

$$y = \frac{dL}{dx},$$

Statuatur iam

$$\int L dx = V$$

functioni cuicunque ipsius x, erit $L = \frac{dV}{dx}$ ideoque

$$\int y dx = \frac{dV}{dx} \quad \text{et} \quad \int yx dx = \frac{xdV}{dx} - V,$$

unde posito elemento dx constante applicata y ita per abscissam x
ut sit $y = \frac{ddV}{dx^2}$.

SCHOLION

25. Me non monente intelligitur, simili modo huiusmodi formul

$$\int YPdx \quad \text{et} \quad \int YQdx$$

ad valores algebraicos reduci posse, si Y functionem quamcunq variabilis y designet, dummodo P et Q sint functiones ipsius x; deter enim ante pro y inventae nunc ipsi Y sunt tribuendae. Quin etiam, si functionem quampiam ipsarum x et y, solutio pari modo absol reductio harum formularum

$$\int Pdx \sqrt{(xx + yy)} \quad \text{et} \quad \int Qdx \sqrt{(xx + yy)}$$

ad valores algebraicos nullam habebit difficultatem, quoniam hae similes evadent propositis, si pro $\sqrt{(xx + yy)}$ scribatur unica litter Unde colligitur ope huius problematis semper binas huiusmodi $\int Vdx$ et $\int Zdx$ ad valores algebraicos perduci posse, quaecunq fuerint functiones ipsarum x et y, dummodo $\frac{V}{Z}$ sit functio ipsius Si enim X sit ista functio, seu $\frac{V}{Z} = X$, loco alterius variabilis y in nova v, ut sit $v = \frac{V}{X}$ seu $v = Z$, atque formulae reducendae erunt

$$\int vXdx \quad \text{et} \quad \int vdx,$$

quarum resolutio iam erit in promtu. Investigemus vero etiam alia fo integralium paria, quae simili modo ad valores algebraicos reduci qu eveniet si quapiam transformatione ad huiusmodi formas revocari

algebraicos adipiscantur.

SOLUTIO

Cum per lemma praemissum sit

$$\int Pdy = Py - \int ydP \quad \text{et} \quad \int Qdy = Qy - \int ydQ$$

quaestio huc redit, ut hae duae formulae integrales $\int ydP$ c
algebraicos consequantur, quod per problema praecedens
efficietur.

I. Statuatur enim

$$\int ydP = L \quad \text{et} \quad \int ydQ = M$$

erit

$$y = \frac{dL}{dP} = \frac{dM}{dQ}, \quad \text{unde fit} \quad \frac{dP}{dQ} = \frac{dL}{dM};$$

ubi cum $\frac{dP}{dQ}$ sit functio ipsius x, si pro L et M functiones qua
cuiusdam variabilis z assumantur, ut $\frac{dL}{dM}$ fiat functio huius
aequatione

$$\frac{dP}{dQ} = \frac{dL}{dM}$$

quantitas x per z determinabitur, ita ut x aequalis reperiatur f
ipsius z. Dehinc aequatio

$$y = \frac{dL}{dP}$$

definiet alteram variabilem y per eandem z; quo facto habebit

$$\int Pdy = \frac{PdL}{dP} - L \quad \text{et} \quad \int Qdy = \frac{QdM}{dQ} - M.$$

II. Pro altera solutione fiat

$$\int ydP = L, \quad \text{ut sit} \quad y = \frac{dL}{dP},$$

eritque altera formula

$$\int ydQ = \int \frac{dQ}{dP} dL = L \cdot \frac{dQ}{dP} - \int Ld \cdot \frac{dQ}{dP};$$

$$\int Ld\cdot\frac{dQ}{dP}=V$$

cuicunque ipsius x, orietur hinc

$$L=\frac{dV}{d(dQ:dP)}.$$

rgo hac quantitate

$$L=\frac{dV}{d(dQ:dP)},$$

functio ipsius x, habebitur altera variabilis

$$y=\frac{dL}{dP}$$

valores algebraici binarum formularum integralium propositarum

$$\int Pdy=Py-L$$

$$\int Qdy=Qy-\frac{LdQ}{dP}+V.$$

COROLLARIUM 1

i hae formulae non debeant esse algebraicae, sed datas quadraturas es, eadem valebunt, quae ad problema praecedens annotavi. Scilicet o debeat esse transcendens, hoc nonnisi per solutionem priorem poterit, sin autem altera tantum quantitatem transcendentem imbeat, per utramque solutionem satisfieri poterit.

COROLLARIUM 2

Hinc etiam patet, si formulae propositae fuerint huiusmodi

$$\int yPdx \quad \text{et} \quad \int Qdy,$$

em ad valores algebraicos pari modo perfici posse. Cum enim sit

$$\int Qdy=Qy-\int ydQ,$$

formulas reduci oportebit

$$\int yPdx \text{ et } \int ydQ,$$

differunt ab iis, quae in praecedente problemate sunt tractatae.

29. Intelligitur etiam, si Y denotet functionem quandam
modo huiusmodi binas formulas

$$\int PYdy \quad \text{et} \quad \int QYdy$$

ad valores algebraicos reduci posse, dummodo $\int Ydy$ integrat
Posito enim

$$\int Ydy = v,$$

formulae reducendae erunt

$$\int Pdv \quad \text{et} \quad \int Qdv,$$

quae hic propositis sunt similes. At si $\int Ydy$ sit functio transc
reductio modo hic exposito non succedit.

PROBLEMA 6

30. Invenire relationem algebraicam inter variabiles x et
formulae integrales

$$\int y^m x^{n-1} dx \quad \text{et} \quad \int y^\mu x^{\nu-1} dx$$

valores algebraicos obtineant.

SOLUTIO

Coaequatis his formulis inter se fit $y^m x^n = y^\mu x^\nu$, unde c
Ponatur ergo

$$y = x^{\frac{\nu - n}{m-\mu}} z,$$

ut sit

$$y^m = x^{\frac{m\nu - mn}{m-\mu}} z^m \quad \text{et} \quad y^\mu = x^{\frac{\mu\nu - \mu n}{m-\mu}} z^\mu$$

atque formulae propositae abibunt in has:

$$\int x^{\frac{m\nu - \mu n}{m-\mu} - 1} z^m dx \quad \text{et} \quad \int x^{\frac{m\nu - \mu n}{m-\mu} - 1} z^\mu dx.$$

Iam vero est:

$$\int x^{\frac{m\nu - \mu n}{m-\mu} - 1} z^m dx = \frac{m-\mu}{m\nu - \mu n} x^{\frac{m\nu - \mu n}{m-\mu}} z^m - \frac{m(m-\mu)}{m\nu - \mu n} \int x^{\frac{m\nu -}{m-}}$$

$$\int x^{\frac{m\nu - \mu n}{m-\mu} - 1} z^\mu dx = \frac{m-\mu}{m\nu - \mu n} x^{\frac{m\nu - \mu n}{m-\mu}} z^\mu - \frac{\mu(m-\mu)}{m\nu - \mu n} \int x^{\frac{m\nu -}{m-\mu}}$$

$$x^{m-\mu} = v,$$

stio perducetur ad has formulas:

$$\int vz^{m-1}dz \quad \text{et} \quad \int vz^{\mu-1}dz,$$

per problema superius sine difficultate resolvuntur.

ALITER

Si neque n neque ν fuerit $= 0$, alia solutio simili modo adhiberi potest. et cum sit

$$\int y^m x^{n-1}dx = \frac{1}{n}y^m x^n - \frac{m}{n}\int x^n y^{m-1}dy \text{ et}$$

$$\int y^\mu x^{\nu-1}dx = \frac{1}{\nu}y^\mu x^\nu - \frac{\mu}{\nu}\int x^\nu y^{\mu-1}dy,$$

stio redit ad has duas formulas:

$$\int x^n y^{m-1}dy \quad \text{et} \quad \int x^\nu y^{\mu-1}dy,$$

posito $x = y^{\frac{\mu-m}{n-\nu}} z$ perinde atque ante tractantur.

COROLLARIUM 1

31. Si sit vel $m = \mu$ vel $n = \nu$, formulae propositae statim per superius ema reduci possunt, sine ulla praevia praeparatione. Casu tamen postequo $n = \nu$ excipiendus est casus quo $n = \nu = 0$; quia reductio supra cripta hic non succedit.

COROLLARIUM 2

32. Per praecepta ergo adhuc tradita huiusmodi binae formulae

$$\int \frac{y^m dx}{x} \text{ et } \int \frac{y^\mu dx}{x}$$

alores algebraicos reduci nequeunt.

COROLLARIUM 3

33. Praeterea vero etiam excipiuntur casus, quibus

$$m\nu = \mu n, \text{ seu } m : n = \mu : \nu,$$

COROLLARIUM 4

34. Sit brevitatis gratia $y^\mu = z$ et $x^\nu = v$, erit $\frac{dx}{x} = \frac{dv}{\nu v}$, unde
irreductibiles sunt

$$\frac{1}{\nu}\int z^\alpha v^{\lambda-1} dv \quad \text{et} \quad \frac{1}{\nu}\int z\, dv.$$

Ac si ulterius ponatur $z = \frac{u}{v}$, hae formulae abibunt in

$$\frac{1}{\nu}\int \frac{u^\alpha dv}{v} \quad \text{et} \quad \frac{1}{\nu}\int \frac{u\, dv}{v},$$

quae iam in formulis Corollarii 2 exclusis continentur.

COROLLARIUM 5

35. Reliquis igitur casibus omnibus, qui in his exceptionib
habent, reductio ad valores algebraicos semper absolvi poterit,
modo pro utraque solutione hic tradita, atque utroque modo gen
valebit secundum binas problematis superioris solutiones.

PROBLEMA 7

36. Si P et Q fuerint functiones ipsius x, invenire relationem
inter x et y, ut ambae hae formulae

$$\int y^m P\, dx \quad \text{et} \quad \int y^n Q\, dx$$

valores algebraicos obtineant.

SOLUTIO

Ponatur

$$y = \left(\frac{Q}{P}\right)^{\frac{1}{m-n}} z \quad \text{seu} \quad y = Q^{\frac{1}{m-n}} P^{\frac{-1}{m-n}} z$$

ex hacque substitutione assequemur:

$$\int y^m P\, dx = \int P^{\frac{-n}{m-n}} Q^{\frac{m}{m-n}} z^m dx,$$

$$\int y^n Q\, dx = \int P^{\frac{-n}{m-n}} Q^{\frac{m}{m-n}} z^n dx.$$

$$\int P^{\frac{-n}{m-n}} Q^{\frac{m}{m-n}} dx$$

egrationem admittat. Nisi enim haec conditio locum habeat, fateor lutionem exhibere non posse. Sit igitur

$$\int P^{\frac{-n}{m-n}} Q^{\frac{m}{m-n}} dz = X$$

eoque X functio algebraica ipsius x, formulaeque reducendae erunt

$$\int z^m dX \quad \text{et} \quad \int z^n dX,$$

ndo resultat

$$\int z^m dX = Xz^m - m\int Xz^{m-1}dz$$

$$\int z^n dX = Xz^n - n\int Xz^{n-1}dz.$$

arum autem formularum reductio supra[1]) iam, idque duplici modo, est osten

COROLLARIUM

37. Si esset $m = n$, problema congrueret cum problemate quarto, ita commoda, quae in hac solutione inde oritura videntur, nihil plane nocea nditio igitur, sub qua reductio propositarum formularum succedit, postu formula differentialis

$$P^{\frac{-n}{m-n}} Q^{\frac{m}{m-n}} dx$$

tegrationem admittat.

PROBLEMA 8

38. Si V et Z sint functiones ipsarum x et y homogeneae, atque V functi mensionum, Z vero functio n dimensionum, invenire relationem algebraic ter x et y, qua duae hae formulae:

$$\int V dx \quad \text{et} \quad \int Z dx$$

ddantur integrabiles.

SOLUTIO

Quia V et Z sunt functiones homogeneae, ita ut ambae variabiles x e bique eundem dimensionum numerum compleant, ibi nempe dimension

1) Vide § 18, 25, 26. H.

$$V = x^m P \text{ et } Z = x^n Q,$$

formulae ad reducendum propositae erunt

$$\int P x^m dx \quad \text{et} \quad \int Q x^n dx,$$

ubi P et Q sunt functiones alterius variabilis t, cuius ad x relationem in
oportet. Iam hae duae formulae ex duabus variabilibus t et x c
reducuntur ad

$$\int P x^m dx = \frac{1}{m+1} P x^{m+1} - \frac{1}{m+1} \int x^{m+1} dP$$

$$\int Q x^n dx = \frac{1}{n+1} Q x^{n+1} - \frac{1}{n+1} \int x^{n+1} dQ,$$

dummodo neque m neque n fuerit $= -1$. Quare cum reductio ad has

$$\int x^{m+1} dP \quad \text{et} \quad \int x^{n+1} dQ$$

revocatur, ponatur

$$x = \left(\frac{dQ}{dP}\right)^{\frac{1}{m-n}} z = z\, dP^{\frac{1}{n-m}} dQ^{\frac{1}{m-n}}$$

formulaeque reducendae erunt

$$\int z^{m+1} dP^{\frac{n+1}{n-m}} dQ^{\frac{m+1}{m-n}} \quad \text{et} \quad \int z^{n+1} dP^{\frac{n+1}{n-m}} dQ^{\frac{m+1}{m-n}},$$

quibus valores algebraicos conciliare licebit, si formula differentialis

$$dP^{\frac{n+1}{n-m}} dQ^{\frac{m+1}{m-n}} = \left(\frac{dP}{dQ}\right)^{\frac{n+1}{n-m}} dQ$$

absolute fuerit integrabilis; reliquis enim casibus haec reductio non
Ponamus ergo hanc formulam esse integrabilem, et cum eius integrale
sit functio algebraica ipsius t, quae sit T, ita ut habeatur

$$\int dP^{\frac{n+1}{n-m}} dQ^{\frac{m+1}{m-n}} = T$$

atque formulae reducendae fient:

$$\int z^{m+1} dT = z^{m+1} T - (m+1) \int T z^m dz$$

$$\int z^{n+1} dT = z^{n+1} T - (n+1) \int T z^n dz.$$

algebraicos obtinere debeant, hoc per problema quartum duplici mod
r.

COROLLARIUM 1

Patet ergo primo, si fuerit vel $m = -1$ vel $n = -1$, reductioner
hodum propositam perfici non posse. Praeterea vero eam quoque locur
ere, nisi formula differentialis

$$dP^{\frac{n+1}{n-m}}dQ^{\frac{m+1}{m-n}}$$

fuerit integrabilis.

COROLLARIUM 2

Quodsi fuerit $m = n$, dummodo utriusque litterae valor non sit $= -1$
ri transformatione non erit opus, sed formulae $\int x^{n+1}dP$ et $\int x^{n+1}dQ$ im
ope problematis quarti reduci poterunt.

EXEMPLUM

Quaeratur relatio algebraica inter x et y, ut hae formulae

$$\int\frac{y^3dx}{xx} \quad et \quad \int\frac{dx}{x^3}(xx+yy)^{\frac{3}{2}}$$

algebraicos obtineant.
a hic sit

$$V = \frac{y^3}{xx} \quad et \quad Z = \frac{1}{x^3}(xx+yy)^{\frac{3}{2}},$$

ergo functio V et Z homogenea, illiusque dimensionum numerus
fuius vero $n = 0$, si ponatur $y = tx$, fiet

$$V = xt^3 \text{ et } Z = (1+tt)^{\frac{3}{2}},$$

lae reducendae erunt

$$\int t^3xdx \quad et \quad \int dx\,(1+tt)^{\frac{3}{2}},$$

$$\int t^3xdx = \tfrac{1}{2}t^3xx - \tfrac{3}{2}\int x^2ttdt$$

$$\int dx\,(1+tt)^{\frac{3}{2}} = x\,(1+tt)^{\frac{3}{2}} - 3\int xtdt\,\sqrt{(1+tt)}.$$

fietque

$$\int x^2 tt dt = \int zz dt (1 + tt) = zz (t + \tfrac{1}{3}t^3) - 2 \int (t +$$

$$\int xt dt \sqrt{(1 + tt)} = \int z dt (1 + tt) = z (t + \tfrac{1}{3}t^3) - \int ($$

Sit brevitatis gratia

$$t + \tfrac{1}{3}t^3 = u,$$

et cum formulae reducendae sint $\int uz dz$ et $\int u dz$, ponatur

$$\int uz dz = L \quad \text{et} \quad \int u dz = M$$

fiet

$$u = \frac{dL}{z dz} = \frac{dM}{dz},$$

ideoque

$$z = \frac{dL}{dM}.$$

Si igitur L et M fuerint functiones quaecunque novae cuiu
aequatio $z = \dfrac{dL}{dM}$ dabit functionem ipsius s pro z, unde etian

$$u = t + \tfrac{1}{3}t^3 = \frac{dM}{dz}$$

dabitur per s; ac propterea pro t reperitur hinc valor in s e
porro dabitur per s variabilis $x = \frac{z}{t} \sqrt{(1 + tt)}$ et $y = tx$, u
et y definiri poterit.

Altera solutio posito

$$\int u dz = L$$

dabit

$$\int uz dz = \int z dL = zL - \int L dz.$$

Sit

$$\int L dz = S$$

existente S functione quacunque ipsius z, fiet

$$L = \frac{dS}{dz};$$

enuo relatio inter x et y reperitur. Nam ob

$$t = \frac{y}{x} \quad \text{et} \quad z = \frac{xy}{\sqrt{(xx + yy)}}$$

res in aequatione

$$\frac{dL}{dz} = \frac{ddS}{dz^2} = \frac{3xxy + y^3}{3x^3}$$

ti dabunt aequationem inter x et y.

PROBLEMA 9

Si V et Z fuerint ut ante functiones homogeneae ipsarum x et y, ill
m, haec vero n dimensionum, invenire relationem algebraicam inte
ut hae duae formulae $\int V dx$ et $\int Z dy$ fiant integrabiles.

SOLUTIO

natur ut ante $y = tx$, fietque $V = x^m P$ et $Z = x^n Q$ existentibus P e
ionibus novae variabilis t, et ob $dy = tdx + xdt$ formulae reducenda

$$\int P x^m dx = \frac{1}{m+1} P x^{m+1} - \frac{1}{m+1} \int x^{m+1} dP$$

$$\int Q x^n dy = \int Q x^n t dx + \int Q x^{n+1} dt;$$

$$\int Q t x^n dx = \frac{1}{n+1} Q t x^{n+1} - \frac{1}{n+1} \int x^{n+1} (Q dt + t dQ),$$

abebimus:

$$\int Q x^n dy = \frac{1}{n+1} Q t x^{n+1} - \frac{1}{n+1} \int x^{n+1} (t dQ - n Q dt).$$

adeo formulae ad valores algebraicos perducendae erunt

$$\int x^{m+1} dP \quad \text{et} \quad \int x^{n+1} (t dQ - n Q dt),$$

onendo

$$x = \left(\frac{t dQ - n Q dt}{dP}\right)^{\frac{1}{m-n}} z$$

$$\left(\frac{udQ - nQdu}{dP}\right)^{\frac{\cdots}{m-n}} dP$$

fuerit integrabilis.

Ubi quidem iterum excludendi sunt casus, quibus v
$n = -1$; praeterea vero notandum est, si sit $m = n$, tum ul
tione ne opus quidem esse, quia formulae $\int x^{m+1} dP$ et $\int x$
statim per problema quartum reduci possunt.

SCHOLION

43. Atque hi sunt fere casus, quibus duae formulae integr
ad valores algebraicos methodo quidem adhuc exposita reduci
autem est dubium, quin haec methodus ad maiorem per
evehi possit, ut etiam formulae hic exclusae ad valores a
queant, quod negotium aliis uberius excolendum relinquo. Co
potissimum casus harum formularum

$$\int \frac{y\,dx}{x} \text{ et } \int \frac{yy\,dx}{x},$$

quas generatim quidem nullo adhuc modo ad integrabilitato
etsi non est difficile innumeras relationes inter x et y exhiber
satisfaciant. His igitur regulis pro duabus formulis prim
contentus, ad tres pluresve formulas eiusdem ordinis progred
turus casus, quibus omnes simul methodo hactenus exposita
braicos reduci queant, quod quidem ea methodo, qua in s
problematis 4 sum usus, praestari debere animadverto.

PROBLEMA 10

44. Si P, Q, R sint functiones quaecunque algebraicae
relationem algebraicam inter variabiles x et y, ut tres hac fo

$$\int yP\,dx, \quad \int yQ\,dx, \quad \int yR\,dx$$

valores algebraicos obtineant.

SOLUTIO

Ponatur

$$\int yP\,dx = L$$

e

$$y = \frac{dL}{P\,dx},$$

duae reliquae formulae reducendae fient:

$$\int yQ\,dx = \int \frac{Q}{P}\,dL = \frac{LQ}{P} - \int L\,d\cdot\frac{Q}{P}$$

$$\int yR\,dx = \int \frac{R}{P}\,dL = \frac{LR}{P} - \int L\,d\cdot\frac{R}{P}.$$

ero hae duae formulae

$$\int Ld\cdot\frac{Q}{P} \quad\text{et}\quad \int Ld\cdot\frac{R}{P}$$

roblema quartum facile resolvuntur, idque duplici modo.

. Priori modo poni oportet:

$$\int Ld\cdot\frac{Q}{P} = M \quad\text{et}\quad \int Ld\cdot\frac{R}{P} = N,$$

uo erit:

$$L = dM : d\cdot\frac{Q}{P} = dN : d\frac{R}{P}.$$

elicitur aequatio

$$\frac{d(Q:P)}{d(R:P)} = \frac{dM}{dN},$$

primum membrum cum sit functio ipsius x, pro M et N capian
ones novae variabilis z, atque per hanc aequationem x definietur i
ssum, unde porro per z dabitur

$$L = \frac{dM}{d(Q:P)} \quad\text{et}\quad y = \frac{dL}{P\,dx}.$$

I. Posteriori resolutione utentes ponamus

$$\int Ld\cdot\frac{Q}{P} = M \quad\text{ut sit}\quad L = \frac{dM}{d(Q:P)},$$

alor in tertia formula substitutus producet

$$\int Ld\cdot\frac{R}{P} = \int dM\cdot\frac{d(R:P)}{d(Q:P)} = M\frac{d(R:P)}{d(Q:P)} - \int Md\cdot\frac{d(R:P)}{d(Q:P)}.$$

ur ergo

$$\int Md\cdot\frac{d(R:P)}{d(Q:P)} = N$$

$$M = \frac{\ldots}{d\cdot\frac{d(R:P)}{d(Q:P)}},$$

unde pro M invenitur functio ipsius x, qua inventa erit

$$L = \frac{dM}{d(Q:P)}$$

ac denique $y = \frac{dL}{Pdx}$. Tum vero valores algebraici trium formular
sitarum erunt:

$$\int yPdx = L$$

$$\int yQdx = \frac{LQ}{P} - M$$

$$\int yRdx = \frac{LR}{P} - M\frac{d(R:P)}{d(Q:P)} + N.$$

COROLLARIUM 1

45. Cum in priori solutione pro litteris M et N functiones q
ipsius z accipi queant, si iis valores transcendentes tribuantur, ita
$\frac{dM}{dz}$ et $\frac{dN}{dz}$ fiant functiones algebraicae, effici poterit, ut trium f
integralium propositarum duae $\int yQdx$ et $\int yRdx$ a datis quadraturis
Quod etiam per problema 2 ita expediri poterit, ut utraque tot qu
casibus nihilominus valores algebraicos adipiscatur.

COROLLARIUM 2

46. Sin autem solutionem posteriorem adhibeamus, quoniam u
N arbitrio nostro relinquitur, si pro ea functio transcendens ipsius x
unius tantum formulae propositae integratio datam quadraturam
reliquae vero duae necessario valores algebraicos obtinebunt.

COROLLARIUM 3

47. Patet etiam, si Y fuerit functio quaecunque ipsius y, simi
tres formulas:

$$\int YPdx, \quad \int YQdx, \quad \int YRdx$$

PROBLEMA 11

48. Si P, Q, R fuerint functiones quaecunque algebraicae variabil venire relationem algebraicam inter x et y, ut hae tres formulae integr

$$\int P dy, \quad \int Q dy, \quad \int R dy$$

lores algebraicos obtineant.

SOLUTIO

Formulae istae per lemma praemissum transformantur in sequentes:

$$\begin{aligned} \int P dy &= Py - \int y dP \\ \int Q dy &= Qy - \int y dQ \\ \int R dy &= Ry - \int y dR. \end{aligned}$$

uaestio ergo redit ad has tres formulas:

$$\int y dP, \quad \int y dQ, \quad \int y dR$$

gebraicas efficiendas, quae cum similes sint iis, quae in problemate praeced nt tractatae, resolutio nullam habebit difficultatem, atque adeo d odo absolvi poterit.

COROLLARIUM 1

49. Quin etiam si ordo inter has formulas immutetur, quoniam per t a quanam earum operatio incipiatur, novem omnino solutiones exh ossunt. Incipiendo enim a prima ponendo $\int y dP = L$, solutio prior adita unam praebet solutionem, posterior vero duas, prout duae rel rmulae sumuntur, vel $\int y dQ$ et $\int y dR$, vel ordine inverso $\int y dR$ et $\int$ eque hinc tres solutiones impetrantur. Atque cum operatio a qualibet h rmularum incohari queat, omnino novem solutiones exhiberi poterunt.

COROLLARIUM 2

50. In hac ergo methodo perinde est, sive formula quaepiam proposi Pdx sive $\int P dy$, quia posterior $\int P dy$ facile ad formam prioris $\int y dP$ tur. Hincque inpostorum nullum amplius discrimen inter duas huius rmulas constituam, ne praeter necessitatem hanc tractationem prolixi ddam.

vel $\int yPdx, \quad \int yQdx, \quad \int Rdy$

vel $\int yPdx, \quad \int Qdy, \quad \int Rdy.$

Superfluum ergo foret diversa hinc problemata constituere.

PROBLEMA 12

52. Ad valores algebraicos reducere quatuor huiusmodi formulas in

$$\int yPdx, \quad \int yQdx, \quad \int yRdx, \quad \int ySdx,$$

in quibus litterae P, Q, R, S denotent functiones quascunque alg
ipsius x.

SOLUTIO

Incipiatur operatio a quacunque harum quatuor formularum
sitarum, ponendo

$$\int yPdx = L,$$

ut sit

$$y = \frac{dL}{Pdx},$$

atque tres reliquae formulae transformabuntur sequenti modo:

$$\int yQdx = \int \frac{Q}{P} dL = \frac{LQ}{P} - \int Ld \cdot \frac{Q}{P}$$

$$\int yRdx = \int \frac{R}{P} dL = \frac{LR}{P} - \int Ld \cdot \frac{R}{P}$$

$$\int ySdx = \int \frac{S}{P} dL = \frac{LS}{P} - \int Ld \cdot \frac{S}{P}.$$

Cum igitur nunc ad valores algebraicos reducendae sint hae tres f

$$\int Ld \cdot \frac{Q}{P}, \ \int Ld \cdot \frac{R}{P}, \ \int Ld \cdot \frac{S}{P}$$

haeque congruant cum iis, quae in problemate 10 sunt pertractatae,
erit in promtu; et quoniam hic novem diversae solutiones suppe
totidemque reperiantur, a quanam alia quatuer formularum prop
initium capiatur, omnino huius problematis quater novem, seu 36 s
exhiberi poterunt.

... ita quaeratur ... propositarum, ... $\int yRdx$, ... atura pendere debeat, ea in operatione ad finem usque est reservan... od 12 modis diversis fieri potest. Sin autem duae formulae datae, vel

$$\int yRdx \quad \text{et} \quad \int ySdx,$$

...tis quadraturis pendere debeant, hoc nonnisi duobus modis diver... tabitur.

COROLLARIUM 2

...54. Hinc etiam patet, eundem solvendi modum ad quinque, pluresq... quot proponantur, similes formulas extendi, dummodo quaelibet form... at speciem

$$\int yPdx \quad \text{vel} \quad \int Pdy,$$

...ente P functione ipsius x, ita ut in singulis formulis altera variabili... si unicam obtineat dimensionem.

COROLLARIUM 3

...55. Quemadmodum in casu duarum huiusmodi formularum propositar... iri possunt 3 solutiones et in casu trium formularum 9 solutiones; sic... 4 formularum inveniuntur $4 \cdot 9 = 36$ solutiones. Atque porro in ca... nularum $5 \cdot 36 = 180$ solutiones, in casu 6 formularum $6 \cdot 180 = 10$... ones, et ita porro.

PROBLEMA 13

...56. Si propositae fuerint quotcunque huiusmodi formulae integrales

$$\int Zdx \quad \text{vel} \quad \int Zdy,$$

...ibus omnibus Z sit functio homogenea ipsarum x et y, et in singulis id... nsionum numerus n deprehendatur; invenire relationem algebraic... x et y, ut singularum harum formularum valores prodeant algebraici.

SOLUTIO

Cum Z sit functio homogenea n dimensionum ipsarum x et y, si pona... x, ea transibit in huiusmodi expressionem $x^n T$, existente T functio... am ipsius t tantum; ideoque quaelibet formula huius generis $\int Z$... itur sequenti modo:

$$dy = tdx + xdt$$

formulae huius generis

$$\int Zdy$$

simili modo transformabuntur:

$$\int Zdy = \int Tx^n (tdx + xdt) = \int x^{n+1} Tdt + \int T$$

at

$$\int Ttx^n dx = \frac{1}{n+1} Ttx^{n+1} - \frac{1}{n+1} \int x^{n+1} (Tdt -$$

unde fiet

$$\int Zdy = \frac{1}{n+1} Ttx^{n+1} - \frac{1}{n+1} \int x^{n+1} (tdT -$$

Quare quotcunque proponantur formulae integrales, vel $\int Zdy$ speciei, quaestio revocabitur ad totidem formulas is

$$\int x^{n+1} \Theta dt,$$

existente Θ functione ipsius t, quae posito $x^{n+1} = u$ abeun

$$\int u \Theta dt.$$

Quotcunque autem huiusmodi formulae $\int u \Theta dt$ fuerint per praecepta hactenus tradita ad valores algebraicos re

COROLLARIUM 1

57. Excipi tamen debent ii casus, quibus function sionum n est $= -1$, seu $n + 1 = 0$, quoniam his ca adhibitae non succedunt.

COROLLARIUM 2

58. Patet etiam, quaecunque et quotcunque fuerin dummodo eae omnes per substitutionem aut transforma huiusmodi formas $\int u \Theta dt$ reduci queant, eas omnes reddi posse.

SCHOLION

59. Vis igitur methodi hactenus expositae in hoc proponantur formulae integrales duas variabiles x et y

singulis altera variabilis y unicam obtineat dimensionem eiusve differen... reductio ad valores algebraicos semper perfici queat; hoc ergo even... gulae formulae fuerint vel huius generis $\int yXdx$, vel huius $\int Xdy$, propt... od huius integratio revocatur ad hanc $\int ydX$, siquidem X sit functio q... que ipsius x. Atque hi sunt casus, quibus duas pluresve formulas integ... mi ordinis mihi quidem adhuc ad valores algebraicos reducere con... ntur vero etiam formulae secundi superiorumque ordinum, quas facil... mulas primi ordinis formae $\int yXdx$ reducere licet, ex quo, si eiusmodi... lae integrales superiorum ordinum occurrant, resolutio problematum ... us allatorum perinde succedet. Eas igitur formulas superiorum ordi... ae huiusmodi reductionem admittunt, hic indicari conveniet.

PROBLEMA 14

60. Si P sit functio quaecunque ipsius x elementumque dx sum... stans, reducere integrationem huiusmodi formularum integralium

$$\int\frac{Pddy}{dx},\quad \int\frac{Pd^3y}{dx^2},\quad \int\frac{Pd^4y}{dx^3}\quad \text{et in genere huius}\quad \int\frac{Pd^ny}{dx^{n-1}}$$

integrationem formulae primi ordinis huiusmodi $\int yQdx$, existente Q f... ne ipsius x.

SOLUTIO

Consideretur formula prima eaque per lemma ita reducetur:

$$\int\frac{Pddy}{dx}=\frac{Pdy}{dx}-\int dy\cdot\frac{dP}{dx}\quad \text{at}\quad \int dy\cdot\frac{dP}{dx}=\frac{ydP}{dx}-\int\frac{yddP}{dx};$$

que erit:

$$\int\frac{Pddy}{dx}=\frac{Pdy}{dx}-\frac{ydP}{dx}+\int\frac{yddP}{dx}.$$

$\frac{ddP}{dx}$ est expressio differentialis formae Qdx, ideoque formula $\int\frac{Pddy}{dx}$ red... ad formulam $\int yQdx$.

Simili modo formula secunda reducitur:

$$\int\frac{Pd^3y}{dx^2}=\frac{Pddy}{dx^2}-\int\frac{dPddy}{dx^2};$$

$$\int\frac{dPddy}{dx^2}=\frac{dPdy}{dx^2}-\frac{yddP}{dx^2}+\int\frac{yd^3P}{dx^2},$$

ubi $\int \frac{yd^3P}{dx^2}$ est iterum formae $\int yQdx$.

Pro tertia formula proposita erit:

$$\int \frac{Pd^4y}{dx^3} = \frac{Pd^3y}{dx^3} - \int \frac{dPd^3y}{dx^3};$$

at per reductionem praecedentem

$$\int \frac{dPd^3y}{dx^3} = \frac{dPddy - dyddP + yd^3P}{dx^3} - \int \frac{yd^4P}{dx^3};$$

ergo

$$\int \frac{Pd^4y}{dx^3} = \frac{Pd^3y - dPddy + dyddP - yd^3P}{dx^3} + \int \frac{yd^4P}{dx^3}$$

ubi iterum $\int \frac{yd^4P}{dx^3}$ est formae $\int yQdx$.

Hinc colligitur fore ulterius progrediendo:

$$\int \frac{Pd^5y}{dx^4} = \frac{Pd^4y - dPd^3y + ddPddy - dyd^3P + yd^4P}{dx^4} - $$

unde etiam generatim patet, hac ratione istius formulae $\int \frac{Pd^n}{dx^{n-}}$
reduci ad integrationem huius formulae $\int \frac{yd^nP}{dx^{n-1}}$, foreque semp
sionem huius formae $\int yQdx$, est enim $\frac{d^nP}{dx^n}$ functio algebraica i
loco si ponatur Q erit

$$\frac{d^nP}{dx^{n-1}} = Qdx.$$

COROLLARIUM 1

61. Omnes ergo reductiones, quae supra circa formulas hu
sunt exhibitae, eodem succedunt modo, si huiusmodi f
proponantur; unde opus non est problemata praecedentia
formulis altiorum ordinum resolvere.

62. Si expressio $\frac{d^n P}{dx^n}$ evanescat, hoc erit indicio, formulam $\int\frac{P d^n y}{dx^n}$ absolute integrabilem; ea ergo his casibus in nostris problematibus locum habebit. Hoc autem evenit, si P fuerit ipsius x huiusmodi functio

$$P = \alpha + \beta x + \gamma x^2 + \delta x^3 + \ldots + \mu x^{n-1}$$

tum enim $\int\frac{P d^n y}{dx^{n-1}}$ integrationem absolute admittet.

COROLLARIUM 3

63. Formulae ergo integrabiles cum suis integralibus erunt pro ipsius n valoribus sequentes:

$$\int \alpha dy = \alpha y$$

$$\int\frac{(\alpha + \beta x)ddy}{dx} = (\alpha + \beta x)\frac{dy}{dx} - \beta y$$

$$\int\frac{(\alpha + \beta x + \gamma x^2)d^3y}{dx^2} = (\alpha + \beta x + \gamma x^2)\frac{ddy}{dx^2} - (\beta + 2\gamma x)\frac{dy}{dx} + 2\gamma y$$

$$\int\frac{(\alpha + \beta x + \gamma x^2 + \delta x^3)d^4y}{dx^3} = (\alpha + \beta x + \gamma x^2 + \delta x^3)\frac{d^3y}{dx^3} - (\beta + 2\gamma x + 3\delta x^2)\frac{ddy}{dx^2}$$
$$+ (2\gamma + 6\delta x)\frac{dy}{dx} - 6\delta y.$$

SCHOLION

64. Progrediamur ergo ad formulas ordinis secundi, cum reductioni earum, quae sunt primi ordinis, iam tantum simus immorati, quantum quidem profectus in hac methodo facti adhuc permiserunt. Quoniam vero ad ordinem secundum eas retulimus formulas, in quibus utriusque variabilis x et y differentialia dx et dy insunt, eae sine dubio sunt simplicissimae, in quibus haec bina differentialia plus una dimensione non obtinent, cuiusmodi in genere est haec formula

$$\int(Vdx + Zdy),$$

ubi V et Z sint functiones quaecunque ipsarum x et y. Nam si unicum adsit differentiale dy, quanquam inde posito $dy = pdx$, littera p in functionem ingreditur, tamen manifestum est, binas variabiles x et y esse commutabiles atque formulas $\int Zdy$ perinde tractari posse, ac $\int Zdx$. Quibus ergo casibus huiusmodi formulis

$$\int(Vdx + Zdy)$$

valores algebraicos conciliare potuerim, explicabo.

algebraicam inter x et y, ut haec formula

$$\int (Vdx + Zdy)$$

algebraicum obtineat valorem.

SOLUTIO

I. Dispiciatur primo, utrum altera pars

$$\int Vdx \quad \text{vel} \quad \int Zdy$$

per lemma reduci possit, ut fiat

vel $$\int Vdx = P - \int Qdy$$
vel $$\int Zdy = R - \int Sdx.$$

Si alterum enim succedit, solutio erit facilis: priori enim casu h

$$\int (Vdx + Zdy) = P + \int (Z - Q)\, dy,$$

posteriori vero

$$\int (Vdx + Zdy) = R + \int (V - S)\, dx.$$

Utravis autem haec formula nullam habet difficultatem per probloma

II. Si hoc modo reductio inveniri nequeat, indagetur functio a
ipsarum x et y, quae sit $= P$, ut

$$\frac{Vdx + Zdy}{P}$$

fiat differentiale functionis cuiuspiam algebraicae Q ipsarum x et y,
casu fiet

$$\int (Vdx + Zdy) = \int PdQ,$$

quae formula nulla difficultate ad integrabilitatem perducitur per pro

III. Saepe etiam huiusmodi functio algebraica ipsarum x et y
inveniri potest, cuius differentiali existente $= Pdx + Qdy$, si ponatu

$$\int (Vdx + Zdy) = T + \int (V - P)\, dx + (Z - Q)\, dy,$$

ut haec formula modo vel primo, vel secundo reductionem admittat.

IV. Interdum quoque iuvabit, in locum unius vel ambarum va
x et y unam duasve novas t et u introducere, ponendis x et y aequali

onibus quibuspiam harum duarum novarum variabilium t et u, ita ut
ubstitutione formula huiusmodi obtineatur

$$\int (Vdx + Zdy) = \int (Pdt + Qdu),$$

bi iam P et Q sunt functiones ipsarum t et u, quae aliquo expositorum n
um reductionem admittat.

V. Casus adhuc singularis est memorandus, quo V et Z sunt funct
omogeneae ipsarum x et y eiusdem ambae numeri dimensionum, qui sit
osito enim $y = tx$ fiet

$$V = Px^n \text{ et } Z = Qx^n,$$

xistentibus P et Q functionibus ipsius t. Tum ob

$$dy = tdx + xdt$$

rmula proposita transibit in hanc

$$\int (Px^n dx + Qtx^n dx + Qx^{n+1} dt),$$

$$\int (P + Qt)\, x^n dx = \frac{1}{n+1} (P + Qt)\, x^{n+1} - \frac{1}{n+1} \int x^{n+1} d\, (P + Qt),$$

ndo reductio revocatur ad huiusmodi formam

$$\int x^{n+1} S dt,$$

si sit $n = -1$, existente S functione ipsius t.

SCHOLION

66. Sufficiat has operationes in genere explicasse, quoniam exempla, q
sum quempiam memorabilem habere videantur, non succurrunt. Inte
men notandum est, plurima exempla proponi posse, quae vel difficu
el plane non, per ullam harum operationum reduci queant. Cuiusmodi
relatio inter x et y quaerenda sit, ut haec formula integralis $\int\left(\frac{ydx}{x} + \right.$
lorem algebraicum obtineat, neque enim video, quomodo huic quaest
tisfaciendum sit. Quamobrem multo minus talia attingo problemata
ibus duae pluresve huiusmodi formulae ad integrabilitatem perduci debe
eque etiam formulas superiorum ordinum generaliter pertractare lice
aeter casum in sequenti problemate contentum.

quantitates finitae x et y in eam non ingrediantur, ad integrabil
hanc formulam $\int Zdx$.

SOLUTIO

Cum formula differentialis Zdx ita sit comparata, ut prae
constantes nonnisi differentialia dx et dy contineat, quae p
dimensionem adimplebunt, cuiusmodi sunt hae formulae:

$$\frac{dy^2}{dx}; \quad \sqrt{(adx^2 + bdxdy + cdy^2)}; \quad \frac{adx^2 + bdy^2}{\sqrt{(dx^2 + dy^2)}} \quad \text{et}$$

ponatur $dy = pdx$, atque formula proposita $\int Zdx$ induet
$\int Pdx$, ita ut P fiat functio quantitatis p tantum, neque x neq
Efficiendum ergo erit, ut non solum haec formula $\int Pdx$, sed etiar
haec $\int pdx$, algebraicum nanciscatur valorem, quod per problen
modo praestabitur. Cum enim sit

$$\int Zdx = \int Pdx = Px - \int xdP$$
$$y = \int pdx = px - \int xdp,$$

fiat primo

$$\int xdP = M \quad \text{et} \quad \int xdp = N$$

eritque

$$x = \frac{dM}{dP} = \frac{dN}{dp},$$

unde fit

$$\frac{dP}{dp} = \frac{dM}{dN},$$

et quia $\frac{dP}{dp}$ est functio ipsius p, inde valor ipsius p erui deb
habebitur

$$x = \frac{dM}{dP} \quad \text{seu} \quad x = \frac{dN}{dp},$$

ac deinceps

$$y = px - N,$$

qui valores praebebunt

$$\int Zdx = Px - M.$$

Pro altera solutione ponatur

$$\int xdP = M,$$

$$\int x dp = \int \frac{dp}{dP}\cdot dM = M\cdot\frac{dp}{dP} - \int M d\cdot\frac{dp}{dP}.$$

onatur $\int M d\cdot\frac{dp}{dP} = R$ functioni ipsius p cuicunque, ac reperietur

$$M = dR : d\cdot\frac{dp}{dP},$$

lore ipsius M invento prodibit porro:

$$x = \frac{dM}{dp};\ y = px - \frac{Mdp}{dP} + R,$$

it

$$\int Z dx = Px - M.$$

natur

$$\int x dp = N.$$

$= \frac{dN}{dp}$ fiet

$$\int x dP = \int dN\cdot\frac{dP}{dp} = N\cdot\frac{dP}{dp} - \int N d\cdot\frac{dP}{dp}.$$

$$\int N d\cdot\frac{dP}{dp} = S,$$

$$N = dS : d\cdot\frac{dP}{dp};$$

e

$$x = \frac{dN}{dp} \text{ et } y = px - N,$$

ons efficitur

$$\int Z dx = Px - \frac{NdP}{dp} + S.$$

COROLLARIUM

. Simili modo solutio exhiberi poterit, si duae pluresve huiusmo
ae $\int Zdx$ proponantur, quibus valores algebraici conciliari debean
enim $dy = pdx$, praeter hanc formulam $\int pdx$, duae pluresve huiu
$\int Pdx$, $\int Qdx$ etc., ubi P et Q etc. sint functiones ipsius p, integrabil
fficiendae, quod per methodos supra traditas facile praestatur.

69. Ut igitur finem huic disquisitioni imponam, eximium e
solvendis praecipuis huius generis problematibus, quae quidem
agitata, ostendam. Versantur autem haec problemata potissimum
rectificabiles algebraicas, quamobrem ex methodis hactenus tr
derivabo regulas, quarum ope tot, quot lubuerit, curvas algebrai
biles reperire liceat, unde simul patebit, quomodo eiusmodi curva
sint inveniendae, quarum integratio a data pendeat quadratura, i
problemata, quae ope cuiuspiam quadraturae sint constructa
rectificationem curvae algebraicae expediri possint. Tum vero n
difficile eiusmodi curvas algebraicas exhibere, quarum rectificatio
data quadratura pendeat, quae tamen nihilo minus unum p
praeciso tot, quot lubuerit, habeant arcus definitos algebraice
Denique solutionem mei illius problematis de duabus curvis, in qu
communi abscissae respondentium summa fiat algebraica, ex l
deducam.

PROBLEMA 17

70. Invenire curvas algebraicas rectificabiles, seu quarum
algebraice exhiberi queant.

SOLUTIO

Sint curvae coordinatae orthogonales x et y, arcusque hi
respondens $= z$. Primo igitur quaeritur aequatio algebraica in
deinde valor ipsius z inde emergens debet esse algebraicus. Cu
$z = \int \sqrt{(dx^2 + dy^2)}$, haec formula integrabilis erit reddenda, qu
bus modis praestabitur.

I. Ponatur $dy = pdx$, atque hae duae formulae

$$y = \int pdx \quad \text{et} \quad z = \int dx\sqrt{(1 + pp)}$$

algebraicae sunt reddendae. Cum igitur sit

$$y = px - \int xdp$$

$$z = x\sqrt{(1 + pp)} - \int \frac{xpdp}{\sqrt{(1 + pp)}},$$

sumantur novae cuiusdam variabilis u functiones quaecunque
P et Q, ponaturque

$$x = \frac{dP}{dp} = \frac{dQ\sqrt{(1+pp)}}{pdp},$$

$$pdP = dQ\sqrt{(1+pp)},$$

$$p = \frac{dQ}{\sqrt{(dP^2 - dQ^2)}}.$$

ergo p per functionem quandam ipsius u, quae ob

$$\frac{dP}{du} \text{ et } \frac{dQ}{du} \text{ ideoque } \frac{dP}{dQ}$$

tes algebraicas, ipsa erit algebraica

$$p = \frac{dQ}{\sqrt{(dP^2 - dQ^2)}},$$

habebitur porro:

$$x = \frac{dP}{dp}, \ y = px - P, \text{ et } z = x\sqrt{(1+pp)} - Q.$$

$$Q = u \text{ et } P = V,$$

posito du constante est

$$dp = \frac{-du\,dV\,ddV}{(dV^2 - du^2)^{\frac{3}{2}}},$$

$$p = \frac{du}{\sqrt{(dV^2 - du^2)}}$$

r:

$$x = \frac{-(dV^2 - du^2)^{\frac{3}{2}}}{du\,ddV}$$

$$y = \frac{-(dV^2 - du^2)}{ddV} - V$$

$$z = \frac{-dV(dV^2 - du^2)}{du\,ddV} - u.$$

ut V sit functio quaecunque ipsius u, ob

$$p = \frac{dV}{\sqrt{(du^2 - dV^2)}} \text{ et } dp = \frac{du^2 ddV}{(du^2 - dV^2)^{\frac{3}{2}}}$$

posito du constante, erit

$$x = \frac{(du^2 - dV^2)^{\frac{3}{2}}}{duddV}$$

$$y = \frac{dV(du^2 - dV^2)}{duddV} - u$$

$$z = \frac{du^2 - dV^2}{ddV} - V.$$

II. Posito ut ante $dy = pdx$, sit

$$\int xdp = M, \text{ ideoque } x = \frac{dM}{dp},$$

unde fit

$$\int \frac{xpdp}{\sqrt{(1+pp)}} = \int \frac{pdM}{\sqrt{(1+pp)}} = \frac{pM}{\sqrt{(1+pp)}} - \int \frac{Mdp}{(1+pp)^{\frac{3}{2}}}$$

Ponatur

$$\int \frac{Mdp}{(1+pp)^{\frac{3}{2}}} = P$$

functioni cuicunque ipsius p, fietque

$$M = \frac{dP}{dp}(1+pp)^{\frac{3}{2}},$$

unde erit porro

$$x = \frac{dM}{dp}, \; y = px - M$$

et

$$z = x\sqrt{(1+pp)} - \frac{Mp}{\sqrt{(1+pp)}} + P.$$

Seu posito dp constante ob

$$dM = \frac{ddP}{dp}(1+pp)^{\frac{3}{2}} + 3pdP\sqrt{(1+pp)}$$

t:

$$x = \frac{ddP}{dp^2}(1+pp)^{\frac{3}{2}} + \frac{3pdP}{dp}\sqrt{(1+pp)}$$

$$y = \frac{pddP}{dp^2}(1+pp)^{\frac{3}{2}} + \frac{(2pp-1)dP}{dp}\sqrt{(1+pp)}$$

$$z = \frac{ddP}{dp^2}(1+pp)^2 + \frac{2p(1+pp)dP}{dp} + P.$$

III. Sit

$$\int \frac{xpdp}{\sqrt{(1+pp)}} = N, \quad \text{erit} \quad x = \frac{dN\sqrt{(1+pp)}}{pdp},$$

oque

$$\int xdp = \int \frac{dN}{p}\sqrt{(1+pp)} = \frac{N}{p}\sqrt{(1+pp)} + \int \frac{Ndp}{pp\sqrt{(1+pp)}}.$$

natur

$$\int \frac{Ndp}{pp\sqrt{(1+pp)}} = P$$

ctioni ipsius p, eritque

$$N = \frac{ppdP\sqrt{(1+pp)}}{dp},$$

quo valore erit porro:

$$= \frac{dN\sqrt{(1+pp)}}{pdp},\ y = px - \frac{N}{p}\sqrt{(1+pp)} - P \text{ et } z = x\sqrt{(1+pp)} -$$

sito autem dp constante ob

$$dN = \frac{ppddP}{dp}\sqrt{(1+pp)} + \left(\frac{pdP(2+3pp)}{\sqrt{(1+pp)}}\right)$$

:

$$x = \frac{pddP(1+pp)}{dp^2} + \frac{dP(2+3pp)}{dp}$$

$$y = \frac{ppddP(1+pp)}{dp^2} + \frac{pdP(1+2pp)}{dp} - P$$

$$z = \frac{pddP(1+pp)^{\frac{3}{2}}}{dp^2} + \frac{2dP(1+pp)^{\frac{3}{2}}}{dp}.$$

IV. Ponatur

$$dy = \frac{dx(qq-1)}{2q}, \quad \text{erit} \quad dz = \frac{dx(qq+1)}{2q}.$$

Hinc fit

$$z + y = \int q\,dx \quad \text{et} \quad z - y = \int \frac{dx}{q};$$

duae ergo hae formulae integrabiles sunt reddendae. Ponatur

$$\int q\,dx = qx - \int x\,dq = qx - M$$

$$\int \frac{dx}{q} = \frac{x}{q} + \int \frac{x\,dq}{qq} = \frac{x}{q} + N,$$

ut sit

$$x = \frac{dM}{dq} = \frac{qq\,dN}{dq};$$

ergo

$$q = \sqrt{\frac{dM}{dN}}.$$

Sint iam M et N functiones quaecunque ipsius u, et ob

$$dq = \frac{dN\,ddM - dM\,ddN}{2\,dN\sqrt{dM\,dN}}$$

erit:

$$x = \frac{2\,dM\,dN\sqrt{dM\,dN}}{dN\,ddM - dM\,ddN}$$

$$z + y = \frac{2\,dM^2\,dN}{dN\,ddM - dM\,ddN} - M$$

$$z - y = \frac{2\,dM\,dN^2}{dN\,ddM - dM\,ddN} + N,$$

ergo

$$y = \frac{dM\,dN\,(dM - dN)}{dN\,ddM - dM\,ddN} - \frac{M + N}{2}$$

et

$$z = \frac{dM\,dN\,(dM + dN)}{dN\,ddM - dM\,ddN} - \frac{M - N}{2}.$$

V. Iisdem positis fiat $\int x\,dq = M$, ut sit

$$\int q\,dx = qx - M,$$

erit

$$x = \frac{dM}{dq}, \quad \text{et} \quad \int \frac{dx}{q} = \frac{x}{q} + \int \frac{dM}{qq} = \frac{x}{q} + \frac{M}{qq} + 2\int \frac{M\,dq}{q^3}.$$

Iam sit

$$\int \frac{M\,dq}{q^3} = Q, \quad \text{ideoque} \quad M = \frac{q^3\,dQ}{dq},$$

$$x = \frac{dM}{dq}, \quad z + y = qx - M \quad \ldots \quad \frac{\ldots}{q} + \frac{\ldots}{qq}$$

$$dM = \frac{q^3 ddQ}{dq} + 3qq dQ,$$

$$x = \frac{q^3 ddQ}{dq^2} + \frac{3qq dQ}{dq}$$

$$z + y = \frac{q^4 ddQ}{dq^2} + \frac{2q^3 dQ}{dq}$$

$$z - y = \frac{qq ddQ}{dq^2} + \frac{4q dQ}{dq} + 2Q$$

ue propterea

$$y = \frac{qq(qq-1)ddQ}{2dq^2} + \frac{q(qq-2)dQ}{dq} - Q$$

$$z = \frac{qq(qq+1)ddQ}{2dq^2} + \frac{q(qq+2)dQ}{dq} + Q.$$

I. Vel fiat

$$\int \frac{x dq}{qq} = N,$$

abeatur

$$x = \frac{qq dN}{dq} \text{ et } \int x dq = \int qq dN = qqN - 2\int Nq dq.$$

ponatur

$$\int Nq dq = Q$$

tente Q functione quacunque ipsius q, atque erit

$$N = \frac{dQ}{q dq}, \; dN = \frac{ddQ}{q dq} - \frac{dQ}{qq},$$

)

$$x = \frac{q ddQ}{dq^2} - \frac{dQ}{dq}; \quad \text{et} \quad \int x dq = \frac{q dQ}{dq} - 2Q,$$

de fiet

$$z + y = \frac{qq ddQ}{dq^2} - \frac{2q dQ}{dq} + 2Q$$

$$z - y = \frac{ddQ}{dq^2}.$$

Quamobrem nanciscemur has formulas

$$x = \frac{qddQ}{dq^2} - \frac{dQ}{dq}$$

$$y = \frac{(qq - 1)\,ddQ}{2dq^2} - \frac{qdQ}{dq} + Q$$

$$z = \frac{(qq + 1)\,ddQ}{2dq^2} - \frac{qdQ}{dq} + Q\,.$$

VII. Ad alias formulas inveniendas ponamus:

$$dx = 2p\,du\,,\ dy = du\,(pp - 1)\ \text{et}\ dz = du\,(pp + 1$$

eritque:

$$x = 2\int p\,du, \quad y + z = 2\int pp\,du, \quad z - y = 2\,u,$$

ergo quaestio ad has duas formulas reducitur:

$$\int p\,du = pu - \int u\,dp, \quad \int pp\,du = ppu - 2\int up\,d$$

Sit nunc

$$\int udp = M \quad \text{et} \quad \int updp = N,$$

erit:

$$u = \frac{dM}{dp} = \frac{dN}{pdp},$$

ideoque

$$p = \frac{dN}{dM} \quad \text{et} \quad dp = \frac{dMddN - dNddM}{dM^2}\,,$$

unde

$$u = \frac{dM^3}{dMddN - dNddM} = \frac{z - y}{2}.$$

Porro est

$$\int p\,du = \frac{x}{2} = \frac{dM^2\,dN}{dMddN - dNddM} - M,$$

et

$$\int ppdu = \frac{z + y}{2} = \frac{dMdN^2}{dMddN - dNddM} - 2\,N;$$

ergo

$$x = \frac{2dM^2dN}{dMddN - dNddM} - 2\,M, \quad y = \frac{dM\,(dN^2 - dM^2)}{dMddN - dNddM}$$

atque

$$z = \frac{dM\,(dN^2 + dM^2)}{dMddN - dNddM} - 2\,N.$$

$$x = \frac{2\,dMdN}{ddN} - 2M$$

$$y = \frac{dN^2 - dM^2}{ddN} - 2N$$

$$z = \frac{dN^2 + dM^2}{ddN} - 2N.$$

VIII. In praecedente solutione ponatur, ut ante

$$\int u dp = M \quad \text{seu} \quad u = \frac{dM}{dp}$$

et

$$\int updp = \int pdM = pM - \int Mdp.$$

am sit

$$\int Mdp = P, \quad \text{erit} \quad M = \frac{dP}{dp} \quad \text{et} \quad dM = \frac{ddP}{dp}$$

ndo fit

$$u = \frac{ddP}{dp^2},$$

tque porro:

$$\tfrac{1}{2}x = \frac{pddP}{dp^2} - \frac{dP}{dp}, \quad \frac{z-y}{2} = \frac{ddP}{dp^2}$$

t

$$\frac{z+y}{2} = \frac{ppddP}{dp^2} - \frac{2pdP}{dp} + 2P$$

hincque eliciuntur istae formulae:

$$x = \frac{2pddP}{dp^2} - \frac{2dP}{dp}$$

$$y = \frac{(pp-1)ddP}{dp^2} - \frac{2pdP}{dp} + 2P$$

$$z = \frac{(pp+1)ddP}{dp^2} - \frac{2pdP}{dp} + 2P.$$

IX. Loco praecedentis operationis fiat

$$\int updp = N, \text{ seu } u = \frac{dN}{pdp},$$

ritque

Iam sit
$$\int \frac{Ndp}{pp} = P,$$
fietque
$$N = \frac{ppdP}{dp} \quad \text{et} \quad dN = \frac{ppddP}{dp} + 2pdP$$
unde
$$u = \frac{pddP}{dp^2} + \frac{2dP}{dp} = \frac{z-y}{2};$$
at erit
$$\frac{z+y}{2} = \frac{p^3ddP}{dp^2} \quad \text{et} \quad \tfrac{1}{2}x = \frac{ppddP}{dp^2} + \frac{pdP}{dp} -$$
ergo
$$x = \frac{2ppddP}{dp^2} + \frac{2pdP}{dp} - 2P$$
$$y = \frac{p(pp-1)ddP}{dp^2} - \frac{2dP}{dp}$$
$$z = \frac{p(pp+1)ddP}{dp^2} + \frac{2dP}{dp}.$$

COROLLARIUM 1

71. Si rectificatio curvae non debeat esse algebraica, se pendere, hoc ope regulae primae ac secundae facile pra enim regula pro V eiusmodi capiatur [posito $P = u$ et Q cendens ipsius u, quae datam quadraturam puta $\int U du$ in $\frac{dV}{du}$ fiat quantitas algebraica, si secunda regula uti volim functio transcendens ipsius p accipi debet.

COROLLARIUM 2

72. Utravis autem regula adhibeatur, id facile expec blematis 2 ut curvae rectificatio indefinita non solum a data sed ut in eadem curva tot, quot lubuerit, extent arcus algebraice exprimi queat.

SCHOLION

73. En ergo novem formulas specie quidem diversas, braicae, rectificabiles, continentur, verumtamen quaolib

vicem reducuntur. Ita solutio quarta ad primam reducitur ponendo

$$M = u + V \quad \text{et} \quad N = u - V.$$

si in sexta ponatur

$$Q = Qqq,$$

itur ad quintam. De his autem solutionibus notandum est, ex singulis
em finitam seu finitis quantitatibus expressam inter tres quantitates
z reperiri posse, cum differentialia inde eliminari queant, pro singulis
olutionibus hac relationes finitae ita se habebunt:

$$
\begin{array}{rl}
\text{I. dat} & (z + V)^2 = x^2 + (y + u)^2 \\
\text{II. dat} & z\sqrt{(1 + pp)} = x + py + P\sqrt{(1 + pp)} \\
\text{III. dat} & z\sqrt{(1 + pp)} = x + py + Pp \\
\text{IV. dat} & (z + y + M)\,(z - y - N) = xx \\
\text{V. dat} & z\,(1 + qq) = 2\,qx + (qq - 1)\,y + 2\,Qqq \\
\text{VI. dat} & z\,(1 + qq) = 2\,qx + (qq - 1)\,y + 2\,Q \\
\text{VII. dat} & (z + y + 4\,N)\,(z - y) = (x + 2\,M)^2 \\
\text{VIII. dat} & (pp + 1)\,z = 2\,px + (pp - 1)\,y + 4\,P \\
\text{IX. dat} & (pp + 1)\,z = 2\,px + (pp - 1)\,y + 4\,Pp
\end{array}
$$

tet solutiones II et III in unam coalescere, si in secunda ponatur

$$P = \frac{R}{\sqrt{(1 + pp)}},$$

ertia

$$P = \frac{R}{p};$$

m prodit haec solutio simplicior:

$$x = \frac{(1 + pp)\,ddR}{dp^2} + \frac{p\,dR}{dp} - R$$

$$y = \frac{p(1 + pp)\,ddR}{dp^2} - \frac{dR}{dp}$$

$$z = \frac{(1 + pp)^{\frac{3}{2}}\,ddR}{dp^2}$$

ut tantum remaneant 4 solutiones quae pro diversis haberi queant:

$$(\mathrm{I},\ \mathrm{IV}),\ (\mathrm{II},\ \mathrm{III}),\ (\mathrm{V},\ \mathrm{VI},\ \mathrm{VIII},\ \mathrm{IX})\ \text{et}\ (\mathrm{VII}).$$

atuor igitur has solutiones principales hic conspectui exponere conven mis earum ita parumper immutatis, ut in singulis sit P functio quaecum ius p.

SOLUTIO I

$$x = \frac{(dp^2 - dP^2)^{\frac{3}{2}}}{dpddP}$$

$$y = \frac{dP(dp^2 - dP^2)}{dpddP} - p$$

$$z = \frac{dp^2 - dP^2}{ddP} - P$$

$$(z + P)^2 = x^2 + (y + p)^2.$$

SOLUTIO II[1]

$$x = \frac{dpdP}{ddP} - p$$

$$y = \frac{dP^2 - dp^2}{2ddP} - P$$

$$z = \frac{dP^2 + dp^2}{2ddP} - P$$

$$(z + P)^2 = (x + p)^2 + (y + P)^2$$

SOLUTIO III

$$x = \frac{(1 + pp)ddP}{dp^2} + \frac{pdP}{dp} - P$$

$$y = \frac{p(1 + pp)ddP}{dp^2} - \frac{dP}{dp}$$

$$z = \frac{(1 + pp)^{\frac{3}{2}}ddP}{dp^2}$$

$$z\sqrt{(1 + pp)} = x + py + P$$

---

1) Haec solutio coalescet in solutionem VII, quae sequuntur coalescent in solutiones II
M

SOLUTIO IV

$$x = \frac{pddP}{dp^2} - \frac{dP}{dp}$$

$$y = \frac{(pp-1)ddP}{2dp^2} - \frac{pdP}{dp} + P$$

$$z = \frac{(pp+1)ddP}{2dp^2} - \frac{pdP}{dp} + P$$

$$(pp+1)z = 2px + (pp-1)y + 2P.$$

igitur, si pro P functiones simpliciores ipsius p substituantur, cur
raicae simpliciores, quae sunt rectificabiles, obtinebuntur, ac paraboli
m ex III erui observo, si ponatur $P = A + Bp^2 + Cp^4 + Dp^6 +$
efficientes debite determinentur.

PROBLEMA 18

4. Invenire duas curvas algebraicas ad eundem axem relatas, quar
sque rectificatio a data quadratura pendeat, ita ut tamen utriusc
m eidem abscissae respondentium summa algebraice exhiberi queat[1]

SOLUTIO

Sit abscissa communis $= x$,

ius curvae applicata $= y$, arcus $= z$;

ltera curva sit applicata $= u$, et arcus $= w$.

tur $dy = pdx$, et $du = qdx$, eritque

pro curva I	pro curva II
$y = px - \int xdp$	$u = qx - \int xdq$
$z = x\sqrt{(1+pp)} - \int \frac{xpdp}{\sqrt{(1+pp)}}$	$w = x\sqrt{(1+qq)} - \int \frac{xqdq}{\sqrt{(1+qq)}}$

sse est ergo primo, ut formulae $\int xdp$ et $\int xdq$ valores nanciscantur al
os, deinde ut summa arcuum $z + w$ sit pariter algebraica, tertio
ue arcus scorsim sumtus, vel, quod eodem redit, arcuum differentia $z -$
a quadratura pendeat.

) Vide L. EULERI Commentationem 48 indicis *Enestroemiani*; p. 76 huius voluminis. H.

$$\sqrt{(1+pp)}-\sqrt{(1+qq)}=s$$

ut sit

$$y=px-\int xdp,\quad u=qx-\int xdq$$

$$z=\frac{x(r+s)}{2}-\tfrac{1}{2}\int x(dr+ds),\quad w=\frac{x(r-s)}{2}-\tfrac{1}{2}\int x(dr-$$

$$z+w=xr-\int xdr$$

$$z-w=xs-\int xds$$

Efficiendum ergo est, ut hae tres formulae:

$$\int xdp,\ \int xdq\ \text{ et }\ \int xdr \text{ fiant algebraicae,}$$

simulque ut formula $\int xds$ a data quadratura pendeat. Ad hoc po

$$\int xdp=L,\ \text{erit}\ x=\frac{dL}{dp},$$

et

$$\int xdq=\int dL\cdot\frac{dq}{dp}=\frac{Ldq}{dp}-\int Ld\cdot\frac{dq}{dp}$$

$$\int xdr=\int dL\cdot\frac{dr}{dp}=\frac{Ldr}{dp}-\int Ld\cdot\frac{dr}{dp}$$

$$\int xds=\int dL\cdot\frac{ds}{dp}=\frac{Lds}{dp}-\int Ld\cdot\frac{ds}{dp}\ .$$

Iam ponatur

$$\int Ld\cdot\frac{dq}{dp}=M,\quad \text{seu}\quad L=\frac{dM}{d\cdot\frac{dq}{dp}}$$

erit

$$\int Ld\cdot\frac{dr}{dp}=\int dM\frac{d\cdot\frac{dr}{dp}}{d\cdot\frac{dq}{dp}}=M\frac{d\cdot\frac{dr}{dp}}{d\cdot\frac{dq}{dp}}-\int Md\cdot\frac{d\cdot\frac{dr}{dp}}{d\cdot\frac{dq}{dp}}$$

et

$$\int Ld\cdot\frac{ds}{dp}=\int dM\frac{d\cdot\frac{ds}{dp}}{d\cdot\frac{dq}{dp}}=M\frac{d\cdot\frac{ds}{dp}}{d\cdot\frac{dq}{dp}}-\int Md\cdot\frac{d\cdot\frac{ds}{dp}}{d\cdot\frac{dq}{dp}}\ .$$

$$\frac{d\cdot\frac{dr}{dp}}{d\cdot\frac{dq}{dp}} = \mu \text{ et } \frac{d\cdot\frac{ds}{dp}}{d\cdot\frac{dq}{dp}} = \nu$$

it

$$\int L d\cdot\frac{dr}{dp} = M\mu - \int M d\mu$$

$$\int L d\cdot\frac{ds}{dp} = M\nu - \int M d\nu.$$

erest, ut formula $\int M d\mu$ reddatur algebraica, altera vero $\int M d\nu$ a da
dratura pendeat. Sit ergo

$$\int M d\mu = N \quad \text{seu} \quad M = \frac{dN}{d\mu},$$

$$\int M d\nu = \int dN\frac{d\nu}{d\mu} = N\frac{d\nu}{d\mu} - \int N d\cdot\frac{d\nu}{d\mu}.$$

iam P eiusmodi functio trancendens, quae datam quadraturam involva
ponatur:

$$\int N d\cdot\frac{d\nu}{d\mu} = P, \text{ ut sit } N = \frac{dP}{d\cdot\frac{d\nu}{d\mu}},$$

valore in praecedentibus formulis substituto reperientur binae curvae alg
iciae quaesito satisfacientes. Sumatur scilicet pro q functio quaecunq
us p, ita ut r et s fiant functiones ipsius p, eruntque etiam μ et ν function
us p; quare pro P capi debebit functio transcendens ipsius p, quae quide
positam quadraturam involvat, hocque modo N dabitur per P, unde dei
s utraque curva definietur. Hinc autem cum $\frac{d\mu}{d\nu}$ sit functio ipsius p, a
ntio exhiberi poterit.

Scilicet ponatur:

$$\int M d\mu = R \quad \text{et} \quad \int M d\nu = S,$$

ut R sit functio algebraica, S vero datam quadraturam includat, eritq

$$M = \frac{dR}{d\mu} = \frac{dS}{d\nu}, \text{ unde fit } \frac{d\mu}{d\nu} = \frac{dR}{dS},$$

utraque curva invenientur.

SCHOLION

75. Haec iam sufficere videntur ad ostendendum quousque
cultura huius novae methodi adhuc pertingere licuit; neque d
specimina aliis ansam sint praebitura, vires suas ad hanc me
promovendam intendendi. Si enim methodus, quae Diophante
quondam ab excellentissimis ingeniis omni studio est exculta,
methodus, quae in quaestionibus longe sublimioribus versatu
tione digna non est aestimanda.

DE AEQUATIONIBUS DIFFERENTIALIBUS SECUNDI GRADUS

Commentatio 265 indicis ENESTROEMIANI
Novi Commentarii academiae scientiarum Petropolitanae 7 (1758/9), 1761, p. 163—202
Summarium ibidem p. 11—12

SUMMARIUM

Singularem atque omnino novam methodum, aequationes differentiales sec adus tractandi, Auctor traditurus, statim observat, plurima atque adeo infinita cap orum evolutio etiamnunc in Mathesi desiderantur, ad Analysin ac potissimum olutionem aequationum differentialium secundi gradus reduci. Quoties enim quae partem quampiam Matheseos, uti vocari solet, applicatae suscipitur, eius enod uabus operationibus absolvitur, quarum alterius ex principiis isti parti propriis sol aequationes analyticas revocatur, altera autem in harum aequationum resolut nsumitur. Iam vero principia Mechanicae, seu Scientiae motus, tam solidorum, q idorum, tam etiam Astronomiae theoreticae, ita sunt exculta, ut vix quaestio exc ri possit, cuius solutionem non istorum principiorum beneficio ad aequationes analyti sque ut plurimum differentiales secundi gradus, perducere liceat. Ex quo manifes ., praecipuam Matheseos perfectionem, quam quidem sperare licet, in huiusr quationum resolutione esse quaerendam. Quam ob causam Cel. Auctor, cum pius in hoc negotio vires suas exercuisset, ac varias methodos particulares, quae sae usum vocari queant, in medium attulisset, hic omnino novam latissimeque paten am ingreditur, istas aequationes tractandi, quae in hoc consistit, ut multiplic vestigetur, in quem huiusmodi aequatio ducta fiat integrabilis: Quin etiam pronunt n dubitat, cuiuscunque fuerit ordinis aequatio differentialis, semper eiusmodi m ontorem negotium conficientem dari, atque in hac dissertatione nonnulla huiusr quationum genera, quae aliis methodis inaccessa videntur, hac methodo feliciter quationes differentiales primi gradus reduxit, neque ullum est dubium, quin haec me s, si uberius excolatur, maxima incrementa in Analysin sit allatura.

quaestio determinatur, ad aequationes analyticas perduc
nem continere dicuntur, altera vero pars in ipsa harum aequ
occupatur. Si quaestio ad Mathesin mixtam, vel applicat
pars petenda est ex principiis, quibus ista disciplina Ma
huicque scientiae quasi est propria; pars autem posterior
puram est referenda, cum tota in resolutione aequation
quaestio, vel ex Mechanica, vel ex Hydrodynamica, vel ex
desumta, ex principiis cuique harum disciplinarum pr
primum ad aequationes reduci oportet, tum vero istarum
lutio artificiis, quae quidem in Analysi comperta habem
quenda. Ex quo satis est manifestum, quanti sit momenti
Matheseos partes.

2. Principia autem fere omnium Matheseos applicat
sunt evoluta, ut nulla propemodum quaestio eo pertinens
solutio non aequationibus comprehendi queat. Sive en
aequilibrio, sive de motu corporum cuiuscunque indolis, ta
fluidorum, cum ab aliis, tum a me, principia certissima su
ope semper ad aequationes pervenire licet: atque si corp
quibuscunque in se invicem agere statuantur, omnes pertu
in eorum motibus efficiuntur, non difficulter ad aequation
si has aequationes resolvere valeremus, nihil amplius su
scientiis desiderari posset. Quocirca omne studium, quod
tur, utilius impendi nequit, quam si in limitibus Ana
elaboremus.

3. Quoties autem problema ad Mathesin applicatam
rarissime in aequationes algebraicas incidimus, quarum re
dum ultra quartum gradum sit perducta, tamen ope a
exacte perfici potest, ut pro perfecta sit habenda. Perpet
vimur ad aequationes differentiales, et quidem maximan
tiales secundi ordinis; principia quippe mechanica statim
gradus implicant: ita ut sine Analyseos infinitorum subsi
scientiis praestari liceat. Cum autem in resolutione ae
tialium primi gradus non admodum simus profecti, mu
dum, si aqua nobis haereat, quando quaestiones ad aequ

nitatae, ut certis tantum casibus, qui non admodum frequenter occurr usum vocari queant. Huiusmodi autem regulas plures exposui in Commenta cademiae Petropolitanae et Volumine VII. Miscellaneorum Berolinensiu

4. Interim tamen iam saepius eiusmodi se mihi obtulerunt casus aeq num differentialium secundi gradus, quas tametsi ope regularum illa nclare non licuerit, tamen aliunde earum integralia habuerim perspe equo ulla via directa patebat, qua haec integralia erui possent. Huiusm asus eo magis sunt notatu digni, quod comparatio illarum aequationum is integralibus tutissimam viam patefacere videatur, earum resolution r certas methodos perficiendi. In quo negotio, si eventus spem non fefell ullum est dubium, quin methodi hunc in finem detectae, multo latius pate : nostram facultatem, aequationes differentiales secundi gradus tracta n mediocriter promoveant. Iis ergo, quos huiusmodi studia iuvant, gratum fore arbitror, si casus illos mihi oblatos commemoravero, ut o onem inde adipiscantur, in hac parte Analysin amplificandi, tum vero ethodos exponam, quas horum casuum contemplatio mihi suppeditavit.

5. Primum huiusmodi exemplum mihi occurrit in Mechanicae meae[2]) T pag. 465, ubi ad hanc perveni aequationem differentialem secundi grad

$$2\,Bxddx - 4\,Bdx^2 = x^{n+6}dp^2\,(1+pp)^{\frac{n-1}{2}},$$

qua differentiale dp sumtum est constans. Eius autem integrale aliu ihi constabat in hac forma contineri[3]):

$$x^{n+5}dp^2\,(1+pp)^{\frac{n+1}{2}} + Cds^3 = 0$$

xistente

$$ds^2 = (1+pp)\,dx^2 + 2\,pxdpdx + xxdp^2.$$

oteram etiam notare valorem huius constantis C esse $= -(n+1)\,B$. um temporis operam inutiliter perdidi in methodo directa indaganda, q

1) L. Euleri Commentationes 10, 62, 188 indicis *Enestroemiani*; vide p. 1, 108, 181 lluminis. H.

2) *Mechanica sive motus scientia analytice exposita*. Tom. I, Petrop. 1736, § 1085. Leon uleri Opera omnia, series II, vol. 1, p. 393. H.

3) Vide § 13. H.

ope istam aequationem integralem ex illa differentiali ... possem, neque ullum artificium cognitum huc deducere ... notari convenit, integrale hic exhibitum tantum esse p... continet quantitatem constantem ab arbitrio nostro p... integrationem esset introducta, infra autem ostendam ... adiici posse huiusmodi terminum Ex^4dp^2.

6. In alind simile exemplum incidi in Opusculorum ... tione[1]) pag. 82, ubi motum corporum in superficiebus m... tatus: perveni autem in evolutione certi cuiusdam casus ... differentialem secundi gradus:

$$\frac{ddr}{r}+\frac{(F+Mkk)^2\,0^2du^2}{(Mkkrr+F+2Gu+Huu)^2}= \ldots$$

ubi differentiale du sumtum est constans, litterae aute... denotant quantitates constantes quascunque. Nullo ... aequationis integrale eruere poteram, aliunde autem n... esse:[2])

$$\frac{(F+Mkk)^2\,0^2du^2}{Mkkrr+F+2Gu+Huu}+\frac{dr^2}{r^2}(F+2Gu+Huu)-\frac{2du\ldots}{r\ldots}$$
$$=\frac{Hdu^2}{r^2}+\frac{(F+Mkk)\,0^2du^2}{rr},$$

quod quidem etiam est particulare, et quia tantopere e... minus patet, quomodo per integrationem ex illa ae... Deinceps vero monstrabo, hoc integrale completum ... $\frac{Hdu^2}{rr}$ adiiciatur $\frac{Cdu^2}{rr}$, ita ut C designet quantitatem ... quae in aequatione differentiali secundi gradus insunt, ...

7. Deinde etiam alia problemata tractans, perdu... aequationes differentiales secundi gradus, quarum int... condita videbatur. Veluti huius aequationis differential...

1) Commentatio 86 indicis *Enestroemiani*: *De motu corporum* ... Opuscula varii argumenti 1, 1746. LEONHARDI EULERI *Opera omnia*, series ...
2) Vide § 23.

to elemento ds constante, integrale particulare quoddam hocce esse[1]):

$$rdr + nrds + nnsds = 0,$$

e quidem aequatio, quia binae variabiles r et s ubique earundem di
um, per methodum a me olim exhibitam[2]), tractari posset. Porro quoq
i obtulit haec aequatio differentio-differentialis:

$$ds^2\,(\alpha ss + \beta s + \gamma) = rrdr^2 + 2\,r^3 ddr$$

to elemento ds constante, cuius integrale completum deprehendi esse

$$C = -\frac{1}{2}\left(\frac{rdr^2}{ds^2} + \frac{\alpha ss + \beta s + \gamma}{r}\right)^2 + \frac{2rdr(2\alpha s + \beta)}{ds} - 2\alpha rr,$$

d, quomodo inde elici queat, haud facile patet. Quin etiam ipsa aeq
gralis, etsi est differentialis primi tantum gradus, parum adiumenti a
tur, ob insignem variabilium implicationem.

8. Haec quatuor exempla sufficiunt, ad ostendendum, plures a
hodos deesse, quibus aequationes differentiales secundi gradus inte
ant, simul autem, quoniam his quidem casibus integralia constan
um inventione non esse desperandum. Equidem post varia tentar
ous has aequationes tractavi, comperi, totum negotium eo redire, ut ic
eratur quantitas, per quam istae aequationes multiplicatae integrati
nittant; tali autem multiplicatore invento, integratio nulla amplius la
cultate. Quemadmodum enim omnium aequationum differentialium
dus integratio eo reduci potest, ut investiganda sit functio quaepiam
variabilium, per quam aequatio multiplicata evadat integrabilis, ita
omnibus aequationibus differentialibus secundi gradus, hanc regulam
ito tanquam generalem in medium afferre, ut statuam semper eius
ctionem variabilium dari, per quam aequatio multiplicata reddatur
bilis.

9. Loquor autem hic de eiusmodi tantum aequationibus, quae
m variabiles involvunt, et quae iam eo sint perductae, ut differen

1) Vide § 35.
2) Confer Commentationes 10 et 44 huius voluminis, imprimis p. 6 et 55.
3) Vide § 40.

aequationes differentiales cuiusque gradus ad formas sequentes
constat:

I. Forma generalis aequationum differentialium primi gr

$$p = \text{funct. } (x \text{ et } y)$$

II. Forma generalis aequationum differentialium secundi g

$$q = \text{funct. } (x,\ y \text{ et } p)$$

III. Forma generalis aequationum differentialium tertii grad

$$r = \text{funct. } (x,\ y,\ p \text{ et } q)$$

IV. Forma generalis aequationum differentialium quarti gra

$$s = \text{funct. } (x,\ y,\ p,\ q \text{ et } r)$$

et ita porro de sequentibus altiorum graduum.

10. Cum igitur proposita quacunque aequatione differentiali $p =$ funct. (x et y), semper detur eiusmodi functio ipsarum x et illa aequatio multiplicata reddatur integrabilis, etiamsi saepe functionem assignare non valeamus, nullum est dubium, qui aequationibus differentialibus secundi gradus $q =$ funct. (x, y et multiplicator existat, qui eas reddat integrabiles, ideoque ad primi gradus reducat. Iam vero hic casus distingui oportet, quib plicator vel binarum tantum variabilium x et y functio existat quantitatem p, seu rationem differentialium $\frac{dy}{dx}$ involvat: ob hoc men ipsa multiplicatoris inventio modo facilior, modo difficilior autem evolutu facillimus habebitur, si multiplicator alterius tant solius fuerit functio.

11. Si igitur litterae P, Q, R, S, T sumantur ad designandas functiones ipsarum variabilium x et y, sequentes ordines simp

Multiplicator ordinis primi	P
Multiplicator ordinis secundi	$Pdx + Qdy$
Multiplicator ordinis tertii	$Pdx^2 + Qdxdy + Rdy^2$
Multiplicator ordinis quarti	$Pdx^3 + Qdx^2dy + Rdxdy^2 +$

etc.

Hi quidem sunt ordines simpliciores, quibus $p = \frac{dy}{dx}$, vel ad nullam ...mam, vel duas, vel tres dimensiones assurgit: facile autem colligitur ...se, ut littera p vel per fractiones, vel irrationalia, vel adeo transcende... ...tiplicatorem afficiat, cuiusmodi casus ingentem campum novarum ...ationum aperiunt. Hic quidem tantum in formis expositis versari cons... ...n eae sufficiunt exemplis allatis expediendis, simulque nos ad aequat... ...to generaliores earum ope resolubiles manuducent.

12. Proposita ergo aequatione quacunque differentiali secundi gr...

$$q = \text{funct.}\ (x,\ y \text{ et } p),$$

...o sumto dx constanti ad hanc formam redigetur

$$ddy = dx^2 \text{ funct.} \left(x,\ y \text{ et } \frac{dy}{dx}\right),$$

...etur primo multiplicator primae formae P, num eius ope integ... ...cedat; sin minus, sumatur multiplicator formae secundae $Pdx + Qdy$ ... negotium conficiat, recurratur ad multiplicatorem formae tertiae, ...rtae, etc.; mox autem colligere licebit, utrum per factores harum form... ...gratio absolvi queat, nec ne; quo posteriori casu ad formas magis co... ...s erit confugiendum, ac dummodo huiusmodi calculo fuerimus as... ...ultatem nobis comparabimus, pro quovis casu oblato idoneam mu...

1) Cf. L. Euleri Commentationes 420, 431, 700 indicis *Enestroemiani*: *De variis integra... ...ribus. Consideratio aequationis differentio-differentialis:*

$$(a + bx)\, ddz + (c + ex)\frac{dxdz}{x} + (f + gx)\,\frac{z\,dx^2}{xx} = 0.$$

...i comment. acad. sc. Petrop. 17, 1773, p. 70; 17, 1773, p. 125. *De formulis differentialibus ...us, quae integrationem admittunt.* Nova acta acad. sc. Petrop. 11, 1798, p. 3. Vide quoque ... ...*um calculi integralis* vol. II § 866—928. *Leonhardi Euleri Opera omnia*, series I, vol. 23 ...

13. Proposita aequatione differentiali secundi gradus:

$$2\,ayddy - 4\,ady^2 - y^{n+5}dx^2\,(1+xx)^{\frac{n}{2}-\frac{1}{2}} = 0,$$

in qua differentiale dx sumtum est constans, eius integrale invenir

SOLUTIO

Factorem primae formae P tentanti mox patebit, negotium non nisi sit $n = -2$, quo quidem casu foret $P = \frac{1}{y^3}$ et aequationis

$$\frac{2ayddy - 4ady^2}{y^3} - \frac{dx^2}{(1+xx)\sqrt{(1+xx)}} = 0$$

integrale esset

$$\frac{2ady}{yy} - \frac{xdx}{\sqrt{(1+xx)}} = adx,$$

denuoque integrando haberetur

$$-\frac{2a}{y} - \sqrt{(1+xx)} = ax + \beta;$$

ita ut hic casus specialis nullam habeat difficultatem. In genere valore quocunque exponentis n, tentetur factor formae secundae et aequatione ad hanc speciem reducta

$$2\,addy - \frac{4ady^2}{y} - y^{n+4}dx^2\,(1+xx)^{\frac{n-1}{2}} = 0$$

productum erit:

$$\left.\begin{array}{l} +\,2aPdxddy - \dfrac{4aPdxdy^2}{y} - Py^{n+4}dx^3\,(1+xx)^{\frac{n-1}{2}} \\ +\,2aQdyddy - \dfrac{4aQdy^3}{y} - Qy^{n+4}dx^2dy\,(1+xx)^{\frac{n-1}{2}} \end{array}\right\} =$$

quam per hypothesin integrabilem esse oportet. Duo autem pri qualescunque P et Q sint functiones ipsarum x et y, nonnisi ex diff horum $2\,aPdxdy + aQdy^2$ oriri potuerunt; unde habebimus *pri integralis* $2\,aPdxdy + aQdy^2$.

$$dP = dx\left(\frac{dP}{dx}\right) + dy\left(\frac{dP}{dy}\right), \quad dQ = dx\left(\frac{dQ}{dx}\right) + dy\left(\frac{dQ}{dy}\right),$$

tio ordinata erit:

$$\left.\begin{array}{r} {}^{n+4}dx^3(1+xx)^{\frac{n-1}{2}} - Qy^{n+4}dx^2dy(1+xx)^{\frac{n-1}{2}} - \frac{4aPdxdy^2}{y} - \frac{4aQdy^3}{y} \\ - 2adx^2dy\left(\frac{dP}{dx}\right) - 2adxdy^2\left(\frac{dP}{dy}\right) - ady^3\left(\frac{dQ}{dy}\right) \\ - adxdy^2\left(\frac{dQ}{dx}\right) \end{array}\right\} =$$

ob dx sumtum constans nullo modo integrabilis esse potest, nisi term
y^3 et dy^2 affecti seorsim se tollant. Necesse ergo est, sit:

$$\frac{4Q}{y} + \left(\frac{dQ}{dy}\right) = 0 \text{ seu } 4Qdy + ydy\left(\frac{dQ}{dy}\right) = 0$$

$$\frac{4P}{y} + 2\left(\frac{dP}{dy}\right) + \left(\frac{dQ}{dx}\right) = 0.$$

ut ex aequatione priori valorem ipsius Q eruamus, spectemus x ut c
oritque

$$dy\left(\frac{dQ}{dy}\right) = dQ,$$

at enim $dy\left(\frac{dQ}{dy}\right)$ incrementum ipsius Q ex solius y variabilitate ortu
cum sit $4Qdy + ydQ = 0$, obtinebimus integrando $Qy^4 = K$ functi
x tantum, ita ut sit

$$Q = \frac{K}{y^4} \text{ et } \left(\frac{dQ}{dx}\right) = \frac{1}{y^4}\left(\frac{dK}{dx}\right),$$

$\left(\frac{dK}{dx}\right)$ erit functio ipsius x. Nunc in altera aequatione quoque x suma
ans, fietque:

$$4Pdy + 2ydP + \frac{dy}{y^3}\left(\frac{dK}{dx}\right) = 0,$$

per y multiplicata et integrata dat:

$$2Pyy - \frac{1}{y}\left(\frac{dK}{dx}\right) = 2L,$$

ideoque

$$P = \frac{L}{yy} + \frac{1}{2y^3}\left(\frac{dK}{dx}\right),$$

ubi L denotat functionem ipsius x tantum. Destructis ergo istis

$$\left(\frac{dP}{dx}\right) = \frac{1}{yy}\left(\frac{dL}{dx}\right) + \frac{1}{2y^3}\left(\frac{ddK}{dx^2}\right)$$

erit *altera pars integralis:*

$$-dx^2\int\left((1+xx)^{\frac{n-1}{2}}\left(Ly^{n+2}dx + \frac{1}{2}y^{n+1}dx\left(\frac{dK}{dx}\right) + Ky^n dy\right)\right) - 2adx^2\int\left(\frac{dy}{yy}\left(\frac{d}{d}\right.\right.$$

quae cum constet duobus membris, pro priori esse debet $L = 0$, c

$$\int(1+xx)^{\frac{n-1}{2}}\left(\frac{1}{2}y^{n+1}dx\left(\frac{dK}{dx}\right) + Ky^n dy\right)$$

integrale erit

$$\frac{Ky^{n+1}}{n+1}(1+xx)^{\frac{n-1}{2}}.$$

Superest ergo, ut reddatur

$$\frac{y^{n+1}dK}{n+1}(1+xx)^{\frac{n-1}{2}} + \frac{(n-1)Ky^{n+1}xdx}{n+1}(1+xx)^{\frac{n-3}{2}} = \frac{1}{2}y^{n+1}dK(1$$

seu

$$2(n-1)Kxdx = (n-1)\ dK\ (1+xx).$$

Atque hinc elicitur $K = 1 + xx$; ita ut alterius partis integra prius sit

$$-\frac{1}{n+1}y^{n+1}dx^2(1+xx)^{\frac{n+1}{2}};$$

at membrum posterius ob $L = 0$ et $\left(\frac{ddK}{dx^2}\right) = 2$ fiet

$$-2adx^2\int\frac{dy}{y^3} = \frac{adx^2}{yy},$$

cuius integratio cum sponte successerit, totum negotium est integralis pars altera erit:

ciendo sit $L=0$ et $K=1+xx$, erit $\left(\frac{dK}{dx}\right)=2x$, hincque fiet:

$$P=\frac{x}{y^3}\quad\text{et}\quad Q=\frac{1+xx}{y^4};$$

integralis pars prima habebitur

$$\frac{2axdxdy}{y^3}+\frac{a(1+xx)dy^2}{y^4}.$$

rca aequationis differentio-differentialis propositae adhibito termin
nte Cdx^2 integrale completum erit:

$$\frac{dx^2}{yy}+\frac{2axdxdy}{y^3}+\frac{a(1+xx)dy^2}{y^4}-\frac{1}{n+1}y^{n+1}dx^2(1+xx)^{\frac{n+1}{2}}=Cdx^2;$$

r y^4 multiplicando:

$$\ldots_{1}\,y^{n+5}dx^2(1+xx)^{\frac{n+1}{2}}=a(yydx^2+2xydxdy+(1+xx)dy^2)-Cy^4dx^2$$

gregio convenit cum eo, quod ante [§ 5] per methodum indirectam era
tus.

COROLLARIUM 1

4. Aequatio ergo differentio-differentialis

$$2addy-\frac{4ady^2}{y}-y^{n+4}dx^2(1+xx)^{\frac{n-1}{2}}=0$$

abilis redditur, si multiplicetur per hunc factorem

$$\frac{xdx}{y^3}+\frac{(1+xx)dy}{y^4},$$

liunde cognosci potuisset, integratio sine ulla difficultate perfecta fuisse

COROLLARIUM 2

5. Vicissim ergo si aequatio integralis inventa

$$\frac{yydx^2+2axydxdy+a(1+xx)dy^2}{y^4}-\frac{1}{n+1}y^{n+1}dx^2(1+xx)^{\frac{n+1}{2}}=Cdx^2$$

$$\frac{xdx}{y^3} + \frac{(1+xx)dy}{y^4},$$

seu hanc

$$xydx + (1+xx)dy,$$

et divisione instituta ipsa demum aequatio differentio-differentiali
proveniet.

COROLLARIUM 3

16. Si aequatio proposita per $\frac{\sqrt{(1+xx)}}{y^4}$ multiplicetur, ut habe

$$2a\left(ddy - \frac{2dy^2}{y}\right)\frac{\sqrt{(1+xx)}}{y^4} - y^n dx^2 (1+xx)^{\frac{n}{2}} = 0,$$

multiplicator eam reddens integrabilem erit:

$$\frac{xydx}{\sqrt{(1+xx)}} + dy\sqrt{(1+xx)} = d\cdot y\sqrt{(1+xx)}.$$

Quare si ponatur

$$y\sqrt{(1+xx)} = z,$$

haec obtinebitur aequatio:

$$\frac{2addz(1+xx)^2}{z^4} - \frac{4adz^2(1+xx)^2}{z^5} + \frac{4axdxdz(1+xx)}{z^4} - \frac{2adx^2}{z^3} -$$

quae per dz multiplicata integrationem admittit. Erit enim integra

$$\frac{adz^2(1+xx)^2}{z^4} + \frac{adx^2}{zz} - \frac{1}{n+1}z^{n+1}dx^2 = Cdx^2.$$

COROLLARIUM 4

17. Hinc ergo patet, quomodo per idoneam substitutionem
sublevari queat; cum enim aequatio proposita per substitutionem y
in hanc posteriorem formam fuerit transmutata, non amplius f
integrationem peragere. Sed praeterquam quod talis substitutio
occurrat, si multiplicator fuerit ordinis tertii, vel altioris, huiusm
ne locum quidem habere poterit.

... In hac solutione usus sum singulari specie calculi, qua ad plu... ...erarum introductionem vitandam differentiale functionis P duarum v... ...um x et y expressi per

$$dP = dx\left(\frac{dP}{dx}\right) + dy\left(\frac{dP}{dy}\right),$$

... more iam satis usitato, $dx\left(\frac{dP}{dx}\right)$ denotat incrementum ipsius P ex ...iabilitate ipsius x oriundum, et $dy\left(\frac{dP}{dy}\right)$ eius incrementum, quod ex v...tate solius y nascitur; constat autem haec duo incrementa addita prae...mplotum differentiale ipsius P ex utra variabili x et y natum. Hinc form...$\left(\frac{dP}{dx}\right)$ et $\left(\frac{dP}{dy}\right)$ denotabunt functiones finitas variabilium x et y, quippe quae ...ferentiationem omissis differentialibus habentur, ita si sit

$$P = y\sqrt{(1 + xx)},$$

...

$$\left(\frac{dP}{dx}\right) = \frac{xy}{\sqrt{(1 + xx)}} \quad \text{et} \quad \left(\frac{dP}{dy}\right) = \sqrt{(1 + xx)}.$$

...m vero cognita altera parte huiusmodi differentialis veluti $dx\left(\frac{dP}{dx}\right)$, ...antitas P inde ex parte cognoscitur. Spectata enim sola x ut variabil...

$$P = \int dx\left(\frac{dP}{dx}\right) + Y,$$

...notante Y functionem ipsius y tantum, atque ex hoc fonte in solu... ...ores quantitatum P et Q determinavi. Manifestum est quoque, si K f...nctio ipsius x tantum, tum $dx\left(\frac{dK}{dx}\right)$ eius completum differentiale iam s...are, ita ut sit $dx\left(\frac{dK}{dx}\right) = dK$; porro autem haec scriptio $\left(\frac{ddK}{dx^2}\right)$ den...m quod $\left(\frac{d\cdot(dK:dx)}{dx}\right)$, seu si ponatur $\left(\frac{dK}{dx}\right) = k$, erit $\left(\frac{ddK}{dx^2}\right) = \left(\frac{dk}{dx}\right)$. Erit ...iter k functio ipsius x tantum; ita si sit $K = \sqrt{(1 + xx)}$, erit

$$\left(\frac{dK}{dx}\right) = \frac{x}{\sqrt{(1 + xx)}} \quad \text{et} \quad \left(\frac{ddK}{dx^2}\right) = \frac{1}{(1 + xx)\sqrt{(1 + xx)}};$$

...equo modo ulterius progredi licebit, ut sit

atque haec ad intelligentiam tam huius solutionis, quam seq
necesse est visum. Caeterum consideratio huius solutionis
sequens Theorema generalius.

THEOREMA 1

19. Ista aequatio differentialis secundi gradus, posito

$$addy - \frac{mady^2}{y} + y^n dx^3 (\alpha + 2\beta x + \gamma xx)^{\frac{n-4m}{2m-}}$$

integrabilis redditur, si multiplicetur per hunc factorem:

$$\frac{(\beta + \gamma x)dx}{(m-1)y^{2m-1}} + \frac{(\alpha + 2\beta x + \gamma xx)dy}{y^{2m}},$$

atque aequatio integralis erit:

$$\frac{\alpha\gamma y^2 dx^2 + 2(m-1)\alpha(\beta + \gamma x)ydxdy + (m-1)^2\alpha(\alpha + 2\beta}{2(m-1)^2 y^{2m}}$$

$$+ \frac{y^{n-2m+1}dx^2}{n-2m+1}(\alpha + 2\beta x + \gamma xx)^{\frac{n-2m+1}{2m-2}} = Cd$$

COROLLARIUM 1

20. Si fuerit $n = 1$, prodibit ista aequatio differential

$$addy - \frac{mady^2}{y} + \frac{ydx^3}{(\alpha + 2\beta x + \gamma xx)^2} = 0,$$

quae ergo multiplicata per

$$\frac{(\beta + \gamma x)dx}{(m-1)y^{2m-1}} + \frac{(\alpha + 2\beta x + \gamma xx)dy}{y^{2m}}$$

fit integrabilis, eius integrali existente:

$$\frac{\alpha\gamma yydx^2 + 2(m-1)\alpha(\beta + \gamma x)ydxdy + (m-1)^2\alpha(\alpha + 2}{2(m-1)^2 y^{2m}}$$

$$- \frac{yydx^2}{2(m-1)y^{2m}(\alpha + 2\beta x + \gamma xx)} = Cdx^2$$

$\ldots = \mu$, si statuamus $y = e^{\int v dx}$, aequatio no
differentialis primi ordinis:

$$adv - \mu avvdx + \frac{dx}{(\alpha + 2\beta x + \gamma xx)^2} = 0\,,$$

cuius ergo integralis erit

$$\alpha\gamma yydx^2 + 2\mu a(\beta + \gamma x)ydxdy + \mu^2 a(\alpha + 2\beta x + \gamma xx)dy^2$$
$$- \frac{\mu yydx^2}{\alpha + 2\beta x + \gamma xx} = 2\mu\mu Cy^{2m}dx^2$$

seu pro y valore suo substituto

$$\alpha\gamma + 2\mu a(\beta + \gamma x)v + \mu^2 a(\alpha + 2\beta x + \gamma xx)vv - \frac{\mu}{\alpha + 2\beta x + \gamma xx} = 2\mu\mu$$

COROLLARIUM 3

22. Statim ergo aequationis differentialis propositae:

$$adv - \mu avvdx + \frac{dx}{(\alpha + 2\beta x + \gamma xx)^2} = 0$$

posito $C = 0$, habemus aequationem integralem particularem, quae

$$0 = \alpha\gamma + 2\mu a(\beta + \gamma x)v + \mu^2 a(\alpha + 2\beta x + \gamma xx)vv - \frac{\mu}{\alpha + 2\beta x +}$$

ex qua per methodum a me alias expositam[1]) integrale completum eru
Quin etiam, si illa aequatio differentialis per hanc formam integralen
tur, integrabilis redditur.

PROBLEMA 2

23. Proposita aequatione differentiali secundi gradus[2]):

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1) L. Euleri Commentatio 95 § 8 et 9; vide p. 167 huius voluminis. Cf. *Instituti integralis*, vol. I, § 541; Commentatio 734, § 4. *Leonhardi Euleri Opera omnia*, series I, vo

2) Pro casu $c = 0$, vide Commentationem 269, § 67, p. 371. Vide quoque *Institution integralis*, vol. II, § 906—910 et Commentationem 734: *Integratio aequationis differentialis*

$$dy + y^2dx = \frac{Adx}{(a + 2bx + cx^2)^2}$$

Mémoires acad. sc. Petersb. 3, 1811, p. 3. *Leonhardi Euleri Opera omnia*, series I, vol. 12 et 2

$$\frac{ddy}{y} + \frac{adx^2}{(a + 2\beta x + \gamma xx + cyy)^2} = 0,$$

in qua differentiale dx sumtum est constans, eius integrale inven

SOLUTIO

Tentetur iterum integratio per factorem $Pdx + Qdy$, ac pos gratia

$$\alpha + 2\,\beta x + \gamma xx + cyy = Z,$$

convertatur aequatio in hanc formam:

$$ddy + \frac{aydx^2}{ZZ} = 0\,,$$

quae per $Pdx + Qdy$ multiplicata praebet:

$$Pdxddy + Qdyddy + \frac{aPydx^3}{ZZ} + \frac{aQydx^2dy}{ZZ} = 0.$$

Quae cum integrabilis esse debeat, dabit statim

I. primam integralis partem $= Pdxdy + \frac{1}{2}Qdy^2$;
superest ergo, ut integrabilis reddatur sequens expressio:

$$-\frac{1}{2}dy^3\left(\frac{dQ}{dy}\right) - \frac{1}{2}dxdy^2\left(\frac{dQ}{dx}\right) + \frac{aQydx^2dy}{ZZ} + \frac{aPydx^3}{ZZ} - dxdy^2\left(\frac{dP}{dy}\right) -$$

Primum ergo necesse est, ut sit $\left(\frac{dQ}{dy}\right) = 0$, unde fit Q functio ips quae sit $Q = K$; tum vero etiam termini dy^2 involventes des ex quibus fit:

$$\left(\frac{dK}{dx}\right) + 2\left(\frac{dP}{dy}\right) = 0$$

seu sumto solo y pro variabili:

$$dy\left(\frac{dK}{dx}\right) + 2dP = 0\ ,$$

cuius integrale est

$$P = L - \frac{1}{2}y\left(\frac{dK}{dx}\right)$$

denotante L quoque functionem ipsius x. Quare ob

$$\left(\frac{dP}{dx}\right) = \left(\frac{dL}{dx}\right) - \frac{1}{2}y\left(\frac{ddK}{dx^2}\right)$$

sumtum constans, altera pars integralis erit:

$$dx^2\int\frac{ay}{ZZ}\left(Ldx-\frac{1}{2}ydx\left(\frac{dK}{dx}\right)+Kdy\right)-dx^2\int dy\left(\left(\frac{dL}{dx}\right)-\frac{1}{2}y\left(\frac{ddK}{dx^2}\right)\right),$$

t

$$\int\frac{aKydy}{ZZ}=aK\int\frac{ydy}{(\alpha+2\beta x+\gamma xx+cyy)^2},$$

pro integrali nascitur

$$II.\ pars=-\frac{a}{2c}\cdot\frac{Kdx^2}{\alpha+2\beta x+\gamma xx+cyy}$$

uo debet esse:

$$\frac{ay}{ZZ}\left(Ldx-\frac{1}{2}ydK\right)=-\frac{a}{2c}\cdot\frac{(\alpha+2\beta x+\gamma xx+cyy)dK-2Kdx(\beta+\gamma x)}{ZZ}$$

$$ydx-\frac{1}{2}acyydK=aKdx(\beta+\gamma x)-\frac{1}{2}adK(\alpha+2\beta x+\gamma xx+cy$$

$$acLydx=aKdx(\beta+\gamma x)-\frac{1}{2}adK(\alpha+2\beta x+\gamma xx).$$

picuum ergo est, esse debere $L=0$ et $K=\alpha+2\beta x+\gamma xx$. Quare
$\Big)=2\gamma$ erit

$$III.\ ultima\ pars\ integralis=+\frac{1}{2}\gamma yydx^2.$$

igitur sit:

$$P=-y(\beta+\gamma x)\ \text{et}\ Q=\alpha+2\beta x+\gamma xx,$$

noster multiplicator:

$$-ydx(\beta+\gamma x)+dy(\alpha+2\beta x+\gamma xx)$$

tegrale quaesitum habebitur:

$$-ydxdy(\beta+\gamma x)+\frac{1}{2}dy^2(\alpha+2\beta x+\gamma xx)-\frac{a(\alpha+2\beta x+\gamma xx)dx^2}{2c(\alpha+2\beta x+\gamma xx+cyy}$$

$$+\frac{1}{2}\gamma yydx^2=Cdx^2.$$

i ponatur $C=\frac{-a}{2c}+C$, erit hoc integrale:

$$+ \frac{yy}{2(\alpha+2\beta x+\gamma xx+cyy)} = Cdx^2.$$

Quae forma convenit cum ea, quam supra [§ 6] exhibui.

THEOREMA 2

24. Ista aequatio differentialis secundi gradus posito dx constan

$$ddy + \frac{ay^{n+1}dx^2}{(\alpha+2\beta x+\gamma xx+cyy)^{\frac{n+4}{2}}} = 0$$

integrabilis redditur per multiplicatorem:

$$-ydx(\beta+\gamma x) + dy(\alpha+2\beta x+\gamma xx)$$

et integrale erit:

$$\frac{1}{2}\gamma yydx^2 - ydxdy(\beta+\gamma x) + \frac{1}{2}dy^2(\alpha+2\beta x+\gamma xx)$$
$$+ \frac{ay^{n+2}dx^2}{(n+2)(\alpha+2\beta x+\gamma xx+cyy)^{\frac{n+2}{2}}} = Cdx^2.$$

COROLLARIUM 1

25. Casus problematis nascitur ex Theoremate hoc, si ponatu Ceterum integrale in Theoremate exhibitum simili modo elicitur, tionem problematis expedivimus; unde superfluum foret, eius demons adiicere.

COROLLARIUM 2

26. Si ponatur $c = 0$, casus habebitur, quem etiam ex Theorem derivare licet, si ibi ponatur $m = 0$. Dum enim pro a scribitur $\frac{1}{a}$ et n, integrale ibi datum perfecte congruit cum hoc, quod istud Theorer ditat pro casu $c = 0$.

COROLLARIUM 3

27. Hoc autem Theorema adeo primum in se complectitur: aequ primi

$$addy - \frac{mady^2}{y} + y^n dx^2(\alpha+2\beta x+\gamma xx)^{\frac{n-4m+3}{2m-2}} = 0,$$

$$\frac{a}{1-m} z^{\frac{m}{1-m}} ddz + z^{\frac{n}{1-m}} dx^2 (\alpha + 2\beta x + \gamma xx)^{\frac{n-4m+3}{2m-2}} = 0$$

$$\frac{addz}{1-m} + z^{\frac{n-m}{1-m}} dx^2 (\alpha + 2\beta x + \gamma xx)^{\frac{n-4m+3}{2m-2}} = 0.$$

iam statuatur $\frac{n-m}{1-m} = \nu + 1$, ut fiat $n = 1 - \nu(m-1)$,

io haec abibit in istam formam:

$$\frac{addz}{1-m} + z^{\nu+1} dx^2 (\alpha + 2\beta x + \gamma xx)^{\frac{-\nu-4}{2}} = 0,$$

st casus particularis praesentis Theorematis, ex quo quippe nascitur, o $c = 0$.

COROLLARIUM 4

. Praesens ergo Theorema latissime patet, atque eiusmodi casus diffi- s in se complectitur, qui nullo alio modo resolvi posse videntur. Si enim fortasse reperietur methodus negotium conficiens, propterea quod les non sunt invicem permixtae: at si c non $= 0$, ob permixtionem varia- nulla methodus cognita hic cum successu in usum vocabitur.

COROLLARIUM 5

. Casus hic imprimis notatu dignus hic occurrit, si $\alpha = 0$, $\beta = 0$, $= 1$, quo habetur haec aequatio:

$$ddy + \frac{ay^{n+1}dx^2}{(xx+yy)^{\frac{n+1}{2}}} = 0,$$

rgo integrale est:

$$\frac{1}{2}(ydx - xdy)^2 + \frac{ay^{n+2}dx^2}{(n+2)(xx+yy)^{\frac{n+2}{2}}} = Cdx^2.$$

r $y = ux$, erit $ydx - xdy = -xxdu$,

$$(n+2)(1+uu)^{\frac{\ }{2}}$$

ideoque

$$\frac{dx}{xx} = \frac{du\,(1+uu)^{\frac{n+2}{4}}}{\sqrt{(2C(1+uu)^{\frac{n+2}{2}} - \frac{2a}{n+2}u^{n+2})}},$$

quae ob variabiles separatas denuo integrari potest.

SCHOLION

30. Hic quoque multiplicatoris forma substitutionem idonea
cuius ope aequatio differentio-differentialis in aliam tractatu
transformabitur. Statui scilicet oportet

$$y = z\sqrt{(\alpha + 2\beta x + \gamma xx)}.$$

Hanc vero ipsam substitutionem suadet formulae indoles

$$(\alpha + 2\beta x + \gamma xx + cyy)^{\frac{n+4}{2}},$$

quia hoc pacto unica variabilis in vinculo relinquitur. At per ha
tionem ipsa aequatio multo magis fit perplexa, ita ut, etiamsi p
simpliciorem

$$dz\,(\alpha + 2\beta x + \gamma xx)^{\frac{3}{2}}$$

ad integrabilitatem revocetur, id tamen minus pateat. Verum si r
fuerit ordinis tertii, seu altioris, ne huiusmodi quidem substitut
inveniri potest, uti in duobus reliquis exemplis usu venit.

PROBLEMA 3

31. Proposita aequatione differentiali secundi gradus:

$$yyddy + mydy^2 = axdx^2,$$

in qua differentiale dx sumtum est constans, eius integrale invo

ertio desumatur. Perducta ergo aequatione ad hanc formam:

$$ddy + \frac{mdy^2}{y} - \frac{axdx^2}{yy} = 0$$

nultiplicetur ea per $Pdx^2 + 2\,Qdxdy + 3\,Rdy^2$, unde statim habebitur:

. *prima pars integralis* $Pdx^2dy + Qdxdy^2 + Rdy^3$

t integrando relinquitur haec forma:

$$-\frac{aPxdx^4}{yy} - \frac{2aQxdx^3dy}{yy} - \frac{3aRxdx^2dy^2}{yy}$$

$$+\frac{mPdx^2dy^2}{y} + \frac{2mQdxdy^3}{y} + \frac{3mRdy^4}{y}$$

$$-dx^3dy\left(\frac{dP}{dx}\right) - dx^2dy^2\left(\frac{dP}{dy}\right) - dxdy^3\left(\frac{dQ}{dy}\right) - dy^4\left(\frac{dR}{dy}\right.$$

$$-dx^2dy^2\left(\frac{dQ}{dx}\right) - dxdy^3\left(\frac{dR}{dx}\right).$$

Haec autem forma integrabilis esse nequit, nisi membra, quae dy^2, dy^3
nplicant, destruantur. Primum ergo pro dy^4 habebimus:

$$\frac{3mR}{y} - \left(\frac{dR}{dy}\right) = 0, \text{ seu } 3\,mRdy = ydR\,,$$

bi x sumitur pro constante, unde fit $R = Ky^{3m}$, denotante K funct
psius x tantum, sicque erit:

$$\left(\frac{dR}{dx}\right) = y^{3m}\left(\frac{dK}{dx}\right).$$

am pro destructione terminorum dy^3 continentium fiet:

$$\frac{2mQ}{y} - \left(\frac{dQ}{dy}\right) - y^{3m}\left(\frac{dK}{dx}\right) = 0$$

eu sumto x constante:

$$2\,mQdy - ydQ = y^{3m+1}dy\left(\frac{dK}{dx}\right),$$

quae divisa per y^{2m+1} et integrata dat:

$$\frac{-Q}{y^{2m}} = \frac{1}{m+1}\,y^{m+1}\left(\frac{dK}{dx}\right) - L$$

sumta denuo L pro functione ipsius x, ita ut sit

$$Q = Ly^{2m} - \frac{1}{m+1} y^{3m+1} \left(\frac{dK}{dx}\right),$$

ideoque

$$\left(\frac{dQ}{dx}\right) = y^{2m} \left(\frac{dL}{dx}\right) - \frac{1}{m+1} y^{3m+1} \left(\frac{ddK}{dx^2}\right).$$

Destruantur denique etiam termini dy^2 continentes, unde prodit:

$$-3aKy^{3m-2}x - y^{2m}\left(\frac{dL}{dx}\right) + \frac{1}{m+1} y^{3m+1}\left(\frac{ddK}{dx^2}\right) + \frac{mP}{y} - \left(\frac{dP}{dy}\right)$$

quae sumta x constante per ydy multiplicata praebet:

$$-3aKxy^{3m-1}dy - y^{2m+1}dy\left(\frac{dL}{dx}\right) + \frac{1}{m+1} y^{3m+2}dy\left(\frac{ddK}{dx^2}\right) + mPdy -$$

quae per y^{m+1} divisa et integrata dat:

$$\frac{-3a}{2m-1} Kxy^{2m-1} - \frac{1}{m+1} y^{m+1}\left(\frac{dL}{dx}\right) + \frac{1}{2(m+1)^2} y^{2m+2}\left(\frac{ddK}{dx^2}\right) - \frac{P}{y^m} +$$

denotante M functionem ipsius x tantum. Ergo fit

$$P = My^m - \frac{3a}{2m-1} Kxy^{3m-1} - \frac{1}{m+1} y^{2m+1}\left(\frac{dL}{dx}\right) + \frac{1}{2(m+1)^2} y^{3m}$$

ideoque

$$\left(\frac{dP}{dx}\right) = y^m\left(\frac{dM}{dx}\right) - \frac{3a}{2m-1} Ky^{3m-1} - \frac{3ax}{2m-1} y^{3m-1}\left(\frac{dK}{dx}\right) - \frac{1}{m+1} y^{2}$$

$$+ \frac{1}{2(m+1)^2} y^{3m+2}\left(\frac{d^3K}{dx^3}\right).$$

Nunc termini

$$-\frac{2aQxdx^3dy}{yy} - dx^3dy\left(\frac{dP}{dx}\right),$$

integrati, x pro constante sumta, suppeditabunt

II. alteram integralis partem:

$$-2axdx^3\left(\frac{1}{2m-1} Ly^{3m-1} - \frac{1}{3m(m+1)} y^{3m}\left(\frac{dK}{dx}\right)\right) - Ndx^3$$

$$-dx^3\left(\frac{1}{m+1} y^{m+1}\left(\frac{dM}{dx}\right) - \frac{a}{m(2m-1)} Ky^{3m} - \frac{ax}{m(2m-1)} y^{3m}\right($$

$$-\frac{1}{2(m+1)^2} y^{3m+2}\left(\frac{ddL}{dx^2}\right) + \frac{1}{6(m+1)^3} y^{3m+3}\left(\frac{d^3K}{dx^3}\right)\Bigg).$$

ius ergo differentiale posito y constante sumtum aequale esse debet res rti $\frac{-aPxdx^4}{yy}$; unde per dx^4 diviso habebimus sequentem aequationem

$$aMxy^{m-2}-\frac{3aaxx}{2m-1}Ky^{3m-3}-\frac{ax}{m+1}y^{2m-1}\left(\frac{dL}{dx}\right)+\frac{ax}{2(m+1)^2}y^{3m}\left(\frac{ddK}{dx^2}\right)$$

$$\frac{2a}{2m-1}Ly^{2m-1}+\frac{2a}{3m(m+1)}y^{3m}\left(\frac{dK}{dx}\right)-\frac{2ax}{2m-1}y^{2m-1}\left(\frac{dL}{dx}\right)+\frac{2ax}{3m(m+1)}y^{3m}$$

$$-\frac{1}{m+1}y^{m+1}\left(\frac{ddM}{dx^2}\right)+\frac{a}{m(2m-1)}y^{3m}\left(\frac{dK}{dx}\right)+\frac{a}{m(2m-1)}y^{3m}\left(\frac{dK}{dx}\right)$$

$$+\frac{ax}{m(2m-1)}y^{3m}\left(\frac{ddK}{dx^2}\right)+\frac{1}{2(m+1)^2}y^{2m+2}\left(\frac{d^3L}{dx^3}\right)-\frac{1}{6(m+1)^3}y^{3m+1}\left(\frac{d^4K}{dx^4}\right)$$

$$=\text{functioni ipsius } x=\left(\frac{dN}{dx}\right).$$

c iam singulae diversae ipsius y potestates scorsim ad nihilum redigant ia y^{m-2} et y^{3m-3} semel occurrunt, nisi sit vel $m=2$, vel $m=1$, habel $=0$ et $K=0$; et supererunt tantum termini per L affecti, inter itarius est y^{2m+2}; unde esse debet $\left(\frac{d^3L}{dx^3}\right)=0$, ideoque

$$L=\alpha+2\beta x+\gamma xx,$$

iqui per y^{2m-1} affecti dant:

$$-\frac{2ax(\beta+\gamma x)}{m+1}-\frac{2a(\alpha+2\beta x+\gamma xx)}{2m-1}-\frac{4ax(\beta+\gamma x)}{2m-1}=0.$$

nc debet esse

$$\alpha=0,\ \text{et}\ \frac{\beta+\gamma x}{m+1}+\frac{4\beta+3\gamma x}{2m-1}=0.$$

ibus conditionibus in genere satisfieri nequit; constituendi ergo sunt quentes:

I. Si $\alpha=0$ et $\gamma=0$, fiet $m=-\frac{1}{2}$, ita ut aequatio proposita sit:

$$yyddy-\tfrac{1}{2}ydy^2=axdx^2$$

1

$$ddy-\frac{dy^2}{2y}-\frac{axdx^2}{yy}=0.$$

m igitur sit $K=0$, $L=x$, $M=0$, erit:

$$R=0,\ Q=\frac{x}{y}\ \text{et}\ P=-2$$

et noster multiplicator erit:

$$-2dx^2 + \frac{2xdxdy}{y}$$

ideoque integrale quaesitum:

$$-2dx^2dy + \frac{xdxdy^2}{y} + \frac{axxdx^3}{yy} = Cdx^3,$$

seu per dx dividendo

$$axxdx^2 + xydy^2 - 2yydxdy = Cyydx^2.$$

II. Sit $\alpha = 0$, $\beta = 0$, erit $m = -\frac{2}{5}$ et aequatio differentio-d
proposita:

$$ddy - \frac{2dy^2}{5y} - \frac{axdx^2}{yy} = 0.$$

Cum igitur sit $K = 0$, $L = xx$ et $M = 0$, erit

$$R = 0,\ \ Q = xxy^{-\frac{4}{5}},\ \ P = -\frac{10}{3}xy^{\frac{1}{5}},$$

unde noster multiplicator fiet:

$$-\frac{10}{3}xy^{\frac{1}{5}}dx^2 + 2xxy^{-\frac{4}{5}}dxdy$$

et integrale quaesitum

$$-\frac{10}{3}xy^{\frac{1}{5}}dx^2dy + xxy^{-\frac{4}{5}}dxdy^2 + \frac{10}{9}ax^3y^{-\frac{9}{5}}dx^3 + \frac{25}{9}y^{\frac{6}{5}}dx^3 =$$

seu per dx dividendo et $y^{\frac{9}{5}}$ multiplicando

$$-\frac{10}{3}xyydxdy + xxydy^2 + \frac{10}{9}ax^3dx^2 + \frac{25}{9}y^3dx^2 = Cy^{\frac{9}{5}}d$$

III. Ante vero iam duos casus commemoravimus, quibus est
vel $m = 2$. Sit ergo primo $m = 1$ et aequatio proposita

$$ddy + \frac{dy^2}{y} - \frac{axdx^2}{yy} = 0$$

ac fieri debet

$$\begin{aligned}\left(\frac{dN}{dx}\right) = \frac{aMx}{y} - 3aaxxK - \frac{1}{2}axy\left(\frac{dL}{dx}\right) + \frac{1}{8}axy^3\left(\frac{ddK}{dx^2}\right) \\ - 2aLy + \frac{1}{3}ay^3\left(\frac{dK}{dx}\right) - 2axy\left(\frac{dL}{dx}\right) + \frac{1}{3}axy^3\left(\frac{ddK}{dx^2}\right) \\ - \frac{1}{2}yy\left(\frac{ddM}{dx^2}\right) + 2ay^3\left(\frac{dK}{dx}\right) + axy^3\left(\frac{ddK}{dx^2}\right) + \frac{1}{8}y^4\left(\frac{d^3L}{dx^3}\right) -\end{aligned}$$

tinemus $M = 0$, $N = -3aa\int Kxxdx$ et

$$x\left(\frac{dL}{dx}\right) - 2L = 0,\ \frac{35}{24}x\left(\frac{ddK}{dx^2}\right) + \frac{7}{3}\left(\frac{dK}{dx}\right) = 0,\ \left(\frac{d^3L}{dx^3}\right) = 0,\ \left(\frac{d^4K}{dx^4}\right) = 0.$$

litionibus satisfit, si sumatur:

$$L = 0,\ \ K = 1,\ \ M = 0\ \text{ et }\ N = -aax^3,$$

: $R = y^3$, $Q = 0$, $P = -3axy^2$.

are noster multiplicator erit:

$$-3axy^2dx^2 + 3y^3dy^2$$

rale quaesitum:

$$-3axy^2dx^2dy + y^3dy^3 + ay^3dx^3 + aax^3dx^3 = Cdx^3.$$

Sit iam $m = 2$, ut aequatio nostra fiat

$$ddy + \frac{2dy^2}{y} - \frac{axdx^2}{yy} = 0,$$

ieri debet huic aequationi:

$$\left(\frac{dN}{dx}\right) = aMx - aaKxxy^3 - \frac{2}{3}aLy^3 - axy^3\left(\frac{dL}{dx}\right) - \frac{1}{3}y^3\left(\frac{ddM}{dx^2}\right)$$
$$+ \frac{4}{9}ay^6\left(\frac{dK}{dx}\right) + \frac{1}{18}y^6\left(\frac{d^3L}{dx^3}\right) + \frac{1}{3}axy^6\left(\frac{ddK}{dx^2}\right) - \frac{1}{162}y^7\left(\frac{d^4K}{dx^4}\right).$$

o $N = a\int Mxdx$, ac statui potest $L = 0$, $K = 0$, $M = 1$, unde fi

xx. Hinc vero fit:

$$R = 0,\ Q = 0,\ P = y^2$$

ultiplicator futurus sit y^2dx^2 et integrale

$$yydx^2dy - \frac{1}{2}axxdx^3 = Cdx^3$$

$$2yydy - axxdx = 2Cdx.$$

COROLLARIUM 1

Casus ergo ultimus, quo $m = 2$, est omnium facillimus, cum per multi

em adeo primi ordinis confici possit, quin primo intuitu aequationi

$$yyddy + 2ydy^2 = axdx^2$$

patet. Casus autem primus et secundus, quibus est $m = -$
multiplicatorem formae secundae, ob $R = 0$, resolvi potu

COROLLARIUM 2

33. Solus ergo casus tertius, quo est $m = 1$, resolutn
requirit multiplicatorem formae tertiae. Quare notetur,
nem differentialem secundi gradus

$$yyddy + ydy^2 - axdx^2 = 0$$

integrabilem reddi, si multiplicetur per $3ydy^2 - 3axdx^2$
et integrale esse:

$$y^3dy^3 - 3axyydx^2dy + ay^3dx^3 + aax^3dx^3$$

COROLLARIUM 3

34. Porro autem notandum est, hanc expressionem
plices resolvi posse. Si enim ponatur brevitatis gratia $a =$
et $\nu = -\frac{1-\sqrt{-3}}{2}$, aequatio haec integralis ita repraese

$$(ydy + cydx + c^2xdx)(ydy + \mu cydx + \nu c^2xdx)(ydy + \nu cy$$

COROLLARIUM 4

35. Hine si constans C sumatur $= 0$, tres statim
integrales particulares:

$$ydy + cydx + c^2xdx = 0$$
$$ydy + \mu cydx + \nu c^2xdx = 0$$
$$ydy + \nu cydx + \mu c^2xdx = 0,$$

quarum prima continet casum iam supra [§ 7] indicatu
sunt imaginariae.

...stat ergo quartum exemplum, quod erat

$$ds^2 (\alpha ss + \beta s + \gamma) = rrdr^2 + 2 r^3 ddr,$$

osito

$$r = y^{\frac{2}{3}}, \text{ ut sit } dr = \frac{2}{3} y^{-\frac{1}{3}} dy \text{ et } ddr = \frac{2}{3} y^{-\frac{1}{3}} ddy - \frac{2}{9} y^{-\frac{4}{3}} dy^2,$$

hanc formam:

$$\frac{4}{3} y^{\frac{5}{3}} ddy = ds^2 (\alpha ss + \beta s + \gamma).$$

ro autem observo, si habeatur huiusmodi aequatio:

$$Sds^2 = mr^\mu dr^2 + nr^{\mu+1} ddr,$$

r substitutionem $r = y^{\frac{n}{m+n}}$ reduci ad hanc formam simpliciorem:

$$Sds^2 = \frac{nn}{m+n} y^{\frac{\mu n - m + n}{m+n}} ddy.$$

nodi ergo aequationes omnes complecti licet in hac forma generali:

$$ddy = y^n X dx^2.$$

ius ergo, quibusnam casibus tam exponentis n, quam functionis X hac o integrari queat per nostram methodum.

PROBLEMA 4

Casus pro exponente n et naturam functionis X invenire, quibus hac o differentialis secundi gradus

$$ddy + y^n X dx^2 = 0,$$

est constans, integrari queat.

SOLUTIO I

matur primo multiplicator primi ordinis P, et integranda erit hac io:

$$Pddy + y^n PX dx^2 = 0,$$

$$y^n PXdx^2 - dxdy\left(\frac{dP}{dx}\right) - dy^2\left(\frac{dP}{dy}\right),$$

unde necesse est, sit $\left(\frac{dP}{dy}\right) = 0$, ideoque P functio ipsius x tantu $P = K$, et integrari oportet ob dx constans:

$$dx\left(y^n KXdx - dy\left(\frac{dK}{dx}\right)\right)$$

cuius integrale nequit esse, nisi

$$-ydx\left(\frac{dK}{dx}\right) = -ydK.$$

Oportet autem sit

$$y^n KXdx^2 + yddK = 0,$$

quod fieri nequit, nisi sub his conditionibus:

$$n = 1 \quad \text{et} \quad X = -\frac{ddK}{Kdx^2},$$

ac tum aequatio integralis erit:

$$Kdy - ydK = Cdx.$$

SOLUTIO II

Sumto multiplicatore secundae formae $Pdx + 2Qdy$, integ cienda est haec aequatio:

$$2Qdyddy + Pdxddy + y^n Xdx^2(Pdx + 2Qdy) = 0,$$

unde *integralis pars prima* colligitur

$$\text{I.} \qquad Pdxdy + Qdy^2.$$

Superest ergo, ut integretur:

$$y^n PXdx^3 + 2y^n QXdx^2dy - dx^2dy\left(\frac{dP}{dx}\right) - dxdy^2\left(\frac{dP}{dy}\right) - dxdy^2\left(\frac{dQ}{dx}\right) - dy^3\left(\frac{dQ}{dy}\right).$$

$\left(\frac{ddK}{dy}\right) = 0$; ideoque $Q = K$ functioni ipsius x.

habebimus

$$\left(\frac{dP}{dy}\right) + \left(\frac{dQ}{dx}\right) = 0, \quad \text{seu} \quad dP + dy\left(\frac{dK}{dx}\right) = 0$$

:

$$P = L - y\left(\frac{dK}{dx}\right) \text{ et } \left(\frac{dP}{dx}\right) = \left(\frac{dL}{dx}\right) - y\left(\frac{ddK}{dx^2}\right).$$

era pars integralis erit:

$$dx^2 \int \left(y^n PX dx + 2y^n QX dy - dy\left(\frac{dP}{dx}\right)\right)$$

$$dx^2 \int \left\{ \begin{matrix} + y^n LX dx & + 2y^n KX dy \\ - y^{n+1} X dx\left(\frac{dK}{dx}\right) - dy\left(\frac{dL}{dx}\right) & + y dy\left(\frac{ddK}{dx^2}\right) \end{matrix} \right\};$$

bilitate ipsius y ergo concluditur *altera pars integralis:*

$$\text{II.} \quad dx^2\left(\frac{2}{n+1} y^{n+1} KX - y\left(\frac{dL}{dx}\right) + \frac{1}{2} yy\left(\frac{ddK}{dx^2}\right) + M\right).$$

bilitas ipsius x postulat, ut sit:

$$y^n LX - y^{n+1} X\left(\frac{dK}{dx}\right) = \frac{2}{n+1} y^{n+1} K\left(\frac{dX}{dx}\right) + \frac{2}{n+1} y^{n+1} X\left(\frac{dK}{dx}\right)$$
$$- y\left(\frac{ddL}{dx^2}\right) + \frac{1}{2} yy\left(\frac{d^3K}{dx^3}\right) + \left(\frac{dM}{dx}\right).$$

velimus indefinitum relinquere, esse debet

$$L = 0, \quad \left(\frac{d^3K}{dx^3}\right) = 0 \quad \text{et} \quad \left(\frac{dM}{dx}\right) = 0;$$

o

$$\frac{2}{n+1} K\left(\frac{dX}{dx}\right) + \frac{n+3}{n+1} X\left(\frac{dK}{dx}\right) = 0,$$

lligitur

$$K^{\frac{n+3}{2}} X = A$$

i; at ob $\left(\frac{d^3K}{dx^3}\right) = 0$ erit

$$K = a + 2\beta x + \gamma xx, \text{ ideoque } X = \frac{A}{(a + 2\beta x + \gamma x ...}$$

et

$$Q = a + 2\beta x + \gamma xx;\ P = -2y(\beta + \gamma x).$$

Quocirca multiplicator erit:

$$-2ydx(\beta + \gamma x) + 2dy(a + 2\beta x + \gamma xx)$$

et huius aequationis differentio-differentialis

$$ddy + \frac{Ay^n dx^2}{(a + 2\beta x + \gamma xx)^{\frac{n+3}{2}}} = 0$$

integrale erit:

$$-2ydxdy(\beta + \gamma x) + dy^2(a + 2\beta x + \gamma xx) + \frac{2}{n+1} \cdot \frac{...}{(a + ...}$$
$$+ \gamma yydx^2 = Cdx^2.$$

Supersunt autem casus, quibus est vel $n = 1$ vel $n = 2$.

I. Sit $n = 1$; et conditiones praecedentes postulant

$$LX + \left(\frac{ddL}{dx^2}\right) = 0;\ \frac{2}{n+1} K\left(\frac{dX}{dx}\right) + \frac{n+3}{n+1} X\left(\frac{dK}{dx}\right) + \frac{1}{2}\left(\frac{d^3 ...}{d ...}\right.$$

seu

$$LXdx^2 + ddL = 0 \text{ et } 2KdX + 4XdK + dx\left(\frac{d^3K}{dx^3}\right)$$

hinc fit

$$2KKX + \int \frac{Kd^3K}{dx^3} = \text{Const.}$$

ideoque

$$2KKXdx^2 + KddK - \frac{1}{2}dK^2 = Edx^2$$

et

$$X = \frac{Edx^2 + \frac{1}{2}dK^2 - KddK}{2KKdx^2}$$

[denotante E constantem]. Pro priori conditione autem pona... erit

$$Q = K,\ P = -y\left(\frac{dK}{dx}\right);$$

atque huius aequationis

$$ddy + yXdx^2 = 0$$

$$X = \frac{\qquad}{2KKdx^2},$$

que functio ipsius x sumatur pro K, erit integrale:

$$-ydxdy\left(\frac{dK}{dx}\right) + Kdy^2 + yyKXdx^2 + \frac{1}{2}yydx^2\left(\frac{ddK}{dx^2}\right) = Cdx^2.$$

Sit $n = 2$; et conditiones postulant:

$$2KdX + 5XdK = 0,\ LX = \frac{1}{2}\left(\frac{d^3K}{dx^3}\right),\ \left(\frac{ddL}{dx^2}\right) = 0.$$

at $X = AK^{-\frac{5}{2}}$, qui in altera substitutus praebet

$$2ALK^{-\frac{5}{2}}dx^3 = d^3K;$$

ob

$$\left(\frac{ddL}{dx^2}\right) = 0, \text{ erit } L = \alpha + \beta x,$$

osito

$$K = (\alpha + \beta x)^\mu$$

$$2A(\alpha + \beta x)^{1-\frac{5\mu}{2}} = \mu(\mu-1)(\mu-2)(\alpha + \beta x)^{\mu-3}\beta^3$$

$\frac{8}{7}$; hincque

$$2A = \frac{-48}{343}\beta^3 \quad \text{et} \quad X = \frac{A}{(\alpha+\beta x)^{\frac{20}{7}}} = \frac{-24\beta^3}{343(\alpha+\beta x)^{\frac{20}{7}}}.$$

$$Q = (\alpha + \beta x)^{\frac{8}{7}};\quad P = \alpha + \beta x - \frac{8}{7}\beta y(\alpha + \beta x)^{\frac{1}{7}}.$$

uenter huius aequationis differentio-differentialis

$$ddy + y^2Xdx^2 = 0$$

to

$$X = \frac{-24\beta^3}{343(\alpha+\beta x)^{\frac{20}{7}}}$$

le est

$$-\beta y dx^3 + \frac{4\beta^2 y^2 dx^2}{49(\alpha+\beta x)^{\frac{6}{7}}} = C dx^2.$$

III. Si $n = 2$, adhuc casus notari meretur, quo $L = a$, et

$$K = x^\mu,$$

erit

$$2aA x^{-\frac{5\mu}{2}} = \mu(\mu-1)(\mu-2)x^{\mu-3},$$

unde fit

$$\mu = \frac{6}{7} \text{ et } 2aA = \frac{6\cdot 1\cdot 8}{343}; \text{ ideoque } a = \frac{24}{343A}.$$

Quare erit

$$K = x^{\frac{6}{7}},\ L = \frac{24}{343A},\ X = \frac{A}{x^{\frac{15}{7}}};$$

ac porro

$$Q = x^{\frac{6}{7}},\quad P = \frac{24}{343A} - \frac{6y}{7x^{\frac{1}{7}}}.$$

Consequenter huius aequationis

$$ddy + \frac{Ay^2dx^2}{x^{\frac{15}{7}}} = 0$$

integrale erit

$$\frac{24dxdy}{343A} - \frac{6ydxdy}{7x^{\frac{1}{7}}} + x^{\frac{6}{7}}dy^2 + \frac{2Ay^3dx^2}{3x^{\frac{9}{7}}} - \frac{3yydx^2}{49x^{\frac{8}{7}}} =$$

SOLUTIO III

Sumto multiplicatore

$$Pdx^2 + 2Qdxdy + 3Rdy^2,$$

prima integralis pars existit

$$Pdx^2dy + Qdxdy^2 + Rdy^3,$$

et reliqua expressio integranda

$$y^n PXdx^4 + 2y^n QXdx^3dy + 3y^n RXdx^2dy^2$$
$$- dx^3dy\left(\frac{dP}{dx}\right) \quad - dx^2dy^2\left(\frac{dP}{dy}\right)$$
$$- dx^2dy^2\left(\frac{dQ}{dx}\right) - dxdy^3\left(\frac{dQ}{dy}\right)$$
$$- dxdy^3\left(\frac{dR}{dx}\right) - dy^4\left(\frac{dR}{dy}\right),$$

e statim, ut ante concludimus, $R = K$ functioni ipsius x, tum vero

$$Q = L - y\left(\frac{dK}{dx}\right), \quad \text{ergo} \quad \left(\frac{dQ}{dx}\right) = \left(\frac{dL}{dx}\right) - y\left(\frac{ddK}{dx^2}\right).$$

nde destructio terminorum per dy^2 affectorum praebet:

$$3y^n KX - \left(\frac{dP}{dy}\right) - \left(\frac{dL}{dx}\right) + y\left(\frac{ddK}{dx^2}\right) = 0,$$

quo fit

$$P = M - y\left(\frac{dL}{dx}\right) + \frac{1}{2}yy\left(\frac{ddK}{dx^2}\right) + \frac{3}{n+1}y^{n+1}KX.$$

n ergo sit

$$\left(\frac{dP}{dx}\right) = \left(\frac{dM}{dx}\right) - y\left(\frac{ddL}{dx^2}\right) + \frac{1}{2}yy\left(\frac{d^3K}{dx^3}\right) + \frac{3}{n+1}y^{n+1}\left(\frac{dKX}{dx}\right),$$

$$2y^n QXdx^3dy = 2Xdx^3\left(y^n Ldy - y^{n+1}dy\left(\frac{dK}{dx}\right)\right)$$

nini per dy affecti praebent *alteram integralis partem*

$$dx^3\left\{\begin{aligned}&\frac{2}{n+1}LXy^{n+1} - \frac{2}{n+2}y^{n+2}X\left(\frac{dK}{dx}\right) - y\left(\frac{dM}{dx}\right) + \frac{1}{2}yy\left(\frac{ddL}{dx^2}\right)\\ &- \frac{1}{6}y^3\left(\frac{d^3K}{dx^3}\right) - \frac{3}{(n+1)(n+2)}y^{n+2}\left(\frac{d.KX}{dx}\right) + N\end{aligned}\right\}.$$

vero, ob primum terminum $y^n PXdx^4$, esse oportet

$$\begin{aligned}0 = y^n MX - y^{n+1}X\left(\frac{dL}{dx}\right) + \frac{1}{2}y^{n+2}X\left(\frac{ddK}{dx^2}\right) + \frac{3}{n+1}y^{2n+1}KXX\\ - \frac{2}{n+1}y^{n+1}\left(\frac{d.LX}{dx}\right) + \frac{2}{n+2}y^{n+2}X\left(\frac{ddK}{dx^2}\right) + \frac{2}{n+2}y^{n+2}\left(\frac{dX}{dx}\right)\left(\frac{dK}{dx}\right)\\ y\left(\frac{ddM}{dx^2}\right) - \frac{1}{2}yy\left(\frac{d^3L}{dx^3}\right) + \frac{1}{6}y^3\left(\frac{d^4K}{dx^4}\right) + \frac{3}{(n+1)(n+2)}y^{n+2}\left(\frac{dd.KX}{dx^2}\right) - \frac{dN}{dx}.\end{aligned}$$

hic casus ad praecedentem deduceretur. Consideremus ergo casu

I. Sit $n = 1$; eritque

$$N = 0, \quad MX + \left(\frac{ddM}{dx^2}\right) = 0;$$

unde ne X ad primam solutionem revocetur, fieri debet $M =$
habebitur:

$$-X\left(\frac{dL}{dx}\right) - \left(\frac{d \cdot LX}{dx}\right) - \frac{1}{2}\left(\frac{d^3L}{dx^3}\right) = 0$$

et

$$\frac{1}{2}X\left(\frac{ddK}{dx^2}\right) + \frac{3}{2}KXX + \frac{2}{3}X\left(\frac{ddK}{dx^2}\right) + \frac{2}{3}\left(\frac{dX}{dx}\right)\left(\frac{dK}{dx}\right) + \frac{1}{6}\left(\frac{d^4K}{dx^4}\right) + \frac{1}{2}$$

Ac ne X ad modum casus praecedentis definiatur, quo erat n
$L = 0$; unde X ex hac aequatione definiri debet:

$$\frac{3}{2}KXXdx^4 + \frac{5}{3}Xdx^2ddK + \frac{5}{3}dx^2dKdX + \frac{1}{2}Kdx^2ddX + $$

II. Sit $n = \frac{1}{2}$; eritque

$$2KXX - \frac{1}{2}\left(\frac{d^3L}{dx^3}\right) = 0, \quad M = 0, \quad N = 0,$$

$$-X\left(\frac{dL}{dx}\right) - \frac{4}{3}\left(\frac{d \cdot Lx}{dx}\right) = 0, \quad \left(\frac{d^4K}{dx^4}\right) = 0;$$

et

$$\frac{13}{10}X\left(\frac{ddK}{dx^2}\right) + \frac{4}{5}\left(\frac{dX}{dx}\right)\left(\frac{dK}{dx}\right) + \frac{4}{5}\left(\frac{dd \cdot KX}{dx^2}\right) = 0,$$

seu

$$\frac{21}{10}XddK + \frac{12}{5}dKdX + \frac{4}{5}KddX = 0,$$

sed huiusmodi casibus non immoror.

SOLUTIO IV

Tentetur etiam factor tertii ordinis

$$Pdx^3 + 2Qdx^2dy + 3Rdxdy^2 + 4Sdy^3,$$

unde nascitur *integralis pars prima:*

$$Pdx^3dy + Qdx^2dy^2 + Rdxdy^3 + Sdy^4$$

reliqua expressio integranda erit:

$$PXdx^5 + 2y^nQXdx^4dy + 3y^nRXdx^3dy^2 + 4y^nSXdx^2dy^3$$
$$- dx^4dy\left(\frac{dP}{dx}\right) - dx^3dy^2\left(\frac{dP}{dy}\right)$$
$$- dx^3dy^2\left(\frac{dQ}{dx}\right) - dx^2dy^3\left(\frac{dQ}{dy}\right)$$
$$- dx^2dy^3\left(\frac{dR}{dx}\right) - dxdy^4\left(\frac{dR}{dy}\right)$$
$$- dxdy^4\left(\frac{dS}{dx}\right) - dy^5($$

it ergo

$$S = K, \quad R = L - y\left(\frac{dK}{dx}\right)$$

que

$$4y^nKXdy - dQ - dy\left(\frac{dL}{dx}\right) + ydy\left(\frac{ddK}{dx^2}\right) = 0.$$

hic in calculos nimis molestos delabamur, ponamus

$$K = A, \quad L = B, \quad \text{ut sit} \quad S = A \quad \text{et} \quad R = B;$$

n ob

$$\left(\frac{dL}{dx}\right) = 0 \quad \text{et} \left(\frac{ddK}{dx^2}\right) = 0, \quad \text{erit} \quad Q = \frac{4A}{n+1}y^{n+1}X.$$

m vero habebimus:

$$3By^nX - \left(\frac{dP}{dy}\right) - \frac{4A}{n+1}y^{n+1}\left(\frac{dX}{dx}\right) = 0,$$

go

$$P = \frac{3}{n+1}BXy^{n+1} - \frac{4A}{(n+1)(n+2)}y^{n+2}\left(\frac{dX}{dx}\right)$$

$$\left(\frac{dP}{dx}\right) = \frac{3B}{n+1}y^{n+1}\left(\frac{dX}{dx}\right) - \frac{4A}{(n+1)(n+2)}y^{n+2}\left(\frac{ddX}{dx^2}\right).$$

ne ergo nascitur *altera integralis pars*:

$$\left(\frac{4A}{(n+1)^2}XXy^{2n+2} - \frac{3B}{(n+1)(n+2)}y^{n+2}\left(\frac{dX}{dx}\right) + \frac{4A}{(n+1)(n+2)(n+3)}y^{n+3}\right($$

oque debet

$$0 = \frac{3B}{n+1} X^2 y^{2n+1} - \frac{4A}{(n+1)(n+2)} X y^{2n+2} \left(\frac{dX}{dx}\right) - \frac{8A}{(n+1)^2} X y^{2 \ldots}$$

$$+ \frac{3B}{(n+1)(n+2)} y^{n+2} \left(\frac{ddX}{dx^2}\right) - \frac{4A}{(n+1)(n+2)(n+3)} y^{n+3} \left(\frac{d^3 \ldots}{dx \ldots}\right.$$

Cui aequationi ut satisfiat, ponatur

$$B = 0 \quad \text{et} \quad \left(\frac{d^3 X}{dx^3}\right) = 0$$

seu

$$X = \alpha + 2\beta x + \gamma xx,$$

fiatque

$$\frac{4A}{(n+1)(n+2)} + \frac{8A}{(n+1)^2} = 0 \quad \text{sive} \quad n = -\frac{5}{3}$$

unde erit:

$S = A, \;\; R = 0, \;\; Q = -6 A y^{-\frac{2}{3}} (\alpha + 2\beta x + \gamma xx) \;$ et $\; P = 36 \ldots$

Quare haec aequatio differentio-differentialis:

$$ddy + y^{-\frac{5}{3}} dx^2 (\alpha + 2\beta x + \gamma xx) = 0$$

fit integrabilis, si multiplicetur per

$$36 y^{\frac{1}{3}} (\beta + \gamma x)\, dx^3 - 12 y^{-\frac{2}{3}} (\alpha + 2\beta x + \gamma xx)\, dx^2 dy + 4 \ldots$$

et integrale erit

$$36 y^{\frac{1}{3}} (\beta + \gamma x)\, dx^3 dy - 6 y^{-\frac{2}{3}} (\alpha + 2\beta x + \gamma xx)\, dx^2 dy^2 + \ldots$$

$$+ 9 y^{-\frac{4}{3}} (\alpha + 2\beta x + \gamma xx)^2\, dx^4 - 27 \gamma y^{\frac{1}{3}} dx^4 = C dx^4 \ldots$$

atque in hac solutione continetur exemplum quartum.

COROLLARIUM 1.

38. Quartum ergo exemplum supra allatum [§ 7 et 36] aequa…
rentialem maxime memorabilem continet, propterea quod ea non…
torem tertii ordinis ad integrabilitatem perduci potest, unde ei…
multo minus ab aliis methodis expectari potest.

). Si vicissim ergo ponamus $y = fz^{\frac{3}{2}}$, ut sit

$$y^{\frac{1}{3}} = z^{\frac{1}{2}} \sqrt[3]{f} \quad \text{et} \quad y^{\frac{5}{3}} = fz^{\frac{5}{2}} \sqrt[3]{ff},$$

$$dy = \frac{3}{2} fz^{\frac{1}{2}} dz \quad \text{et} \quad ddy = \frac{3}{2} fz^{\frac{1}{2}} ddz + \frac{3}{4} fz^{-\frac{1}{2}} dz^2$$

uatio proposita:

$$\frac{3}{2} fz^{\frac{1}{2}} ddz + \frac{3}{4} fz^{-\frac{1}{2}} dz^2 + \frac{dx^2(\alpha + 2\beta x + \gamma xx)}{fz^{\frac{5}{2}} \sqrt[3]{ff}}$$

egrabilis, si multiplicetur per

$$36 z^{\frac{1}{2}} (\beta + \gamma x) dx^3 \sqrt[3]{f} - \frac{18(\alpha + 2\beta x + \gamma xx) dx^2 dz}{z^{\frac{1}{2}}} \sqrt[3]{f} + \frac{27}{2} f^3 z^{\frac{3}{2}} dz^3$$

grale erit:

$$4fz(\beta + \gamma x) dx^3 dz \sqrt[3]{f} - \frac{27}{2} f(\alpha + 2\beta x + \gamma xx) dx^2 dz^2 \sqrt[3]{f} + \frac{81}{16} f^4 zz dz^4$$
$$+ \frac{9(\alpha + 2\beta x + \gamma xx)^2 dx^4}{fzz \sqrt[3]{f}} - 27\gamma fzz dx^4 \sqrt[3]{f} = C dx^4.$$

COROLLARIUM 3

. Ponatur $ff \sqrt[3]{ff} = \frac{4}{3}$, ut habeatur haec aequatio:

$$2z^3 ddz + zz dz^2 + dx^2 (\alpha + 2\beta x + \gamma xx) = 0,$$

c fiet integrabilis, si multiplicetur per:

$$\frac{2(\beta + \gamma x) dx^3}{zz} - \frac{(\alpha + 2\beta x + \gamma xx) dx^2 dz}{z^3} + \frac{dz^3}{z},$$

integrale:

$$4z(\beta + \gamma x) dx^3 dz - (\alpha + 2\beta x + \gamma xx) dx^2 dz^2 + \frac{1}{2} zz dz^4$$
$$+ \frac{(\alpha + 2\beta x + \gamma xx)^2 dx^4}{2zz} - 2\gamma zz dx^4 = C dx^4,$$

equatio etiam hoc modo repraesentari potest:

$$2\beta x + \gamma xx) dx^2 - zz dz^2)^2 + 8z^3 (\beta + \gamma x) dx^3 dz - 4\gamma z^4 dx^4 = E zz dx^4$$

41. Si sit $\alpha = 0, \beta = 0$ et $\gamma = a^2$, seu ista aequatio integran

$$2z^3ddz + zzdz^2 + aaxxdx^2 = 0,$$

ea integrabilis reddetur per hunc multiplicatorem:

$$\frac{2aaxdx^3}{zz} - \frac{aaxxdx^2dz}{z^3} + \frac{dz^3}{z}$$

et aequatio integralis erit:

$$(aaxxdx^2 - zzdz^2)^2 + 8aaxz^3dx^3dz - 4aaz^4dx^4 = E$$

seu

$$(aaxxdx^2 + zzdz^2)^2 - 4aa(zdx - xdz)^2 zzdx^2 = Ez$$

COROLLARIUM 5

42. Posita ergo constante $E = 0$, pro hoc casu gemina aeq particularis habebitur:

$$\text{I. } aaxxdx^2 + zzdz^2 - 2azdx(zdx - xdz) = 0$$
$$\text{II. } aaxxdx^2 + zzdz^2 + 2azdx(zdx - xdz) = 0$$

quarum illa resolvitur in

$$axdx + zdz = \pm zdx\sqrt{2a}$$

haec vero in

$$axdx - zdz = \pm zdx\sqrt{-2a}.$$

SCHOLION

43. Evolutio horum exemplorum ita est comparata, ut non in resolutione aequationum differentialium secundi gradus a cum enim haec exempla, si nonnullos casus faciliores excipiamu rum adhuc usitatarum expediri nequeant, nova haec methodu per multiplicatores conficitur, non solum optimo cum success etiam nullum est dubium, quin ea, si uberius excolatur, multo sit allatura. Pari autem quoque successu ad aequationes di et altiorum graduum extendi poterit, siquidem certum est, q

s. Quod cum etiam verum sit in aequationibus differentialibus primi gradu
arum resolutio per methodum tales factores investigandi non mediocrite
moveri poterit; ubi quidem totum negotium eo reducitur, ut quovis cas
ato idoneus multiplicator inveniatur; atque in aequationibus quide
ferentialibus primi gradus hic factor semper erit functio ipsarum x et
tum, verum ob hoc ipsum quod diversitas ordinum locum non habet, ei
estigatio multo difficilior videtur, imprimis quando iste factor est funct
nscendens. Cum autem haec ratio integrandi naturae aequationum s
ximo consentanea, non sine eximio fructu studium in ea excolenda coll
itur.

DE INTEGRATIONE AEQUATIONUM DIFFERENTIALIUM

Commentatio 269 indicis ENESTROEMIANI
Novi Commentarii academiae scientiarum Petropolitanae 8 (1760/1), 1763,
Summarium ibidem p. 5—12

SUMMARIUM

Saeculum mox erit elapsum, ex quo idea Differentialium et Integral successu in Analysin est invecta, unde etiam haec scientia tanta subito menta, ut, quicquid antea fuerat exploratum, vix comparationem sustinea autem hoc novum calculi genus summorum ingeniorum studio et indefess est excultum, minime tamen id exhaustum est reputandum, et quo ulteriu penetrare licet, eo ampliores campos etiam nunc prorsus incultos detegu qui vires humanas longe superare videntur. Cum igitur labores in hoc stud tantum utilitatis attulerint, eo magis hinc animi Geometrarum inflamm omnibus viribus immensum hunc campum perscrutari annitantur. Quorum antiquis tantum elementis sunt adstricti, vel qui a Mathematicis dis abhorrent, eos idea Infiniti, cui subliniores istae investigationes sunt sup mediocriter offendere solet, et voce perperam intellecta, plerumque s subtiliorem hanc Analyseos partem tantum in vanis circa Infinite magna c speculationibus consumi, neque inde quicquam utilitatis ad vera cog obiecta, quippe quae omnia sint finita, expectari posse. Quae opinio, etsi u tis, quae sublimiori Analysi accepta referre debemus, iam funditus es tamen abs re erit perversas illas Infiniti ideas, quibus ea innititur, remove Cum igitur universa Mathesis in omnis generis quantitatum contemplatio tione versetur, nemo ignorat, plerasque quantitates, quas in mundo continuo variari, et perpetuis mutationibus esse obnoxias. Coelum inspici solem, lunam et stellas situm suum iugiter mutare, sola illa stella except in ipso mundi polo fixa apparet: situm autem per quantitates cognoscim cuiusque stellae, sive respectu nostri Horizontis per Altitudinem et A

onomiam cognitione quantitatum contineri, quarum aliae continuas mutat antur, modo maiores modo minores, aliae vero perpetuo eaedem maneant, tudo cuiusque stellae fixae, etiamsi nunc quidem hic levis variatio sit observata ergo quantitatum, quas natura nobis offert, divisio in Variabiles et Constantes nanifesta, simulque intelligitur, difficillimam nostrae cognitionis partem in acc ntitatum Variabilium investigatione esse constitutam. Scilicet tum demum perfe itionem motuum coelestium, veluti planetae, seu cometae, sumus adepti, cum vis tempore eius locum in coelo, hoc est, eius Longitudinem et Latitudinem, assi erimus. Ponamus nobis lunae motum hac ratione esse exploratum, quo melius no ationes figere queamus; quicquid enim de hoc casu dixero, id facile ad omnis g ntitates variabiles transferetur. Cum igitur ad quodvis tempus, quod pariter quan mur, lunae tam Longitudo, quam Latitudo, assignari queat, utraque haec quan tempus determinatur, seu si tempus a certa epocha elapsum denotetur littera t gitudo, quam Latitudo lunae exprimetur certa quadam formula per tempus t uten ita, cuius valorem pro quovis tempore t assignare liceat. Huiusmodi formula gene s valor determinatus pro quolibet tempore determinato exhiberi potest, vocat ysi Functio quantitatis t, sicque nostro casu et Longitudo et Latitudo luna quaedam *Functio temporis* t, cuius *natura*, hoc est *ratio compositionis*, si nobis pecta, motus lunae perfectam haberemus cognitionem, quae igitur tota in ra um functionum sita est censenda. Cum igitur inde constet, quantam mutationem gitudo et Latitudo quovis tempore subeat, variatio etiam, minimo tempore facta, ae et ipsa erit minima, definiri, eiusque relatio ad ipsum tempus minimum assi rit; quae cognitio maximi est momenti, cum inde mutatio momentanea innot que hic impedit, quo minus tempusculum istud evanescens seu infinite parvum acc Atque hic est fons *Infinito parvorum, in Analysi receptorum; ubi probe notari conv* tam ipsorum parvitatem, quam rationem mutuam, quae utique est finita, conside emadmodum huiusmodi Infinite parva Differentialia vocantur, ita Calculus, in e ione scrutanda occupatus, Differentialis appellatur: neque hic quicquam de In is est metuendum, cum omnis calculus in eorum relatione, quae est finita, absolv tenus quidem assumsimus indolem earum formularum, seu Functionum, quae L nem lunae et Latitudinem per tempus exprimunt, esse cognitam; verum si vic mutatio *momentanea daretur, quippe quam ex viribus lunam sollicitantibus col* , tum quaestio ad naturam illarum Functionum investigandam reducitur, tot o theoria ipsi est superstruenda. Hic igitur ex mutatione momentanea, seu, ut ae loquuntur, ex data relatione Differentialium, indoles ac natura ipsarum fu determinari debet, in quo Calculus Integralis continetur. Quemadmodum it ulus Differentialis docet Functionum Differentialia, seu potius eorum rationen gare, ita vicissim Calculus Integralis ex data Differentialium ratione indolem Fu eruendi methodum *tradit.* Utriusque ergo vim ita commodissime describere v fuerit Functio quaecunque quantitatis t, ac ponatur Differentialium ratio $\frac{dv}{dt}$ ulus Differentialis methodum exhibeat, ex indole Functionis v hanc Differentia

inde natura Functionis v, seu quomodo ea per t determinetur,
ex illa aequatione data quantitatem $p = \frac{dv}{dt}$ per t et v definire liceo

$$Mdt + Ndv = 0$$

nascetur, Differentialis appellata, in qua litterae M et N utcunqu
sunt intelligendae, et iam quaeritur, cuiusmodi functio quantita
eodem redit, aequatio relationem inter t et v exprimens requiritu
ipsius t valor ipsius v assignari queat.

Hanc igitur quaestionem in latissimo sensu acceptam Col
tatione contemplatur, et postquam animadvertit, eam tantum p
resolvi posse, atque in hunc finem methodos maxime diversas a C
methodum multo simpliciorem magisque naturalem exponit, om
quae simul viam ad plurimos alios casus patefacere videtur. Qua
ex ipso Auctoris scripto est iudicandum; hic tantum notasse iu
$Mdt + Ndv = 0$ etiam in latissimo sensu acceptam, exiguam
versae Analyseos infinitorum continere, quia tantum Different
plectitur. Quodsi enim v fuerit functio quaecunque ipsius t, et Diff
$\frac{dv}{dt} = p$, etiam haec quantitas p est variabilis, ex cuius variatio
potest $\frac{dp}{dt} = q$, quae quantitas q Differentialia secundi ordinis
cum pariter a t pendeat, ponaturque $\frac{dq}{dt} = r$, haec littera Differe
plicare censetur, et ita porro. Quibus positis Calculus Integralis
methodus ex data relatione Differentialium cuiusque ordinis i
vestigandi, ex qua illa Differentialia nascantur, seu, quod eodem
quaecunque inter quantitates t, v, p, q, r etc. quemadmodum quan
investigari oportet. Ab hoc autem perfectionis gradu omnia a
artificia multo magis sunt remota, et quae adhuc ignorantur, imm
illam particulam, quam etiamnum evolvere licuit.

Verum ne sic quidem tota vis Analysis infinitorum exhaurit
functiones hic sumus contemplati, quae per unicam variabile
longitudo vel latitudo lunae spectari poterat tanquam Functio
qua tempus exprimitur. Dantur autem utique casus, quibus oc
runtur, quae simul per binas, vel ternas, vel adeo plures variab

Huiusmodi exemplum se offert, quando motus fluvii definir
tatem pro omnibus punctis, quae in fluvio concipere licet, deter
autem puncti situs per ternas coordinatas x, y et z definitur, et oc
tanquam Functio ternarum istarum variabilium x, y et z erit c
relatio inter harum et ipsius Functionis quaesitae Differentialia
quam forte ex principiis motus colligere licet, quaestio huc red

$$dv = p\,dx + q\,dy + r\,dz,$$

ata relatione inter quantitates v, x, y, z, p, q, r, aequatione quacunque expressa, q quomodo functio v per variabiles x, y et z exprimatur. Tum vero, cum etiam rae sint functiones coordinatarum x, y et z, earum quoque Differentialia, quae se nis sunt censenda, in computum ingredi possunt, unde hanc quaestionem, ut lati eat, etiam ad relationem Differentialium secundi altiorumque ordinum extendi et. Quodsi motus fluminis etiam cum tempore varietur, tum ad eius cogniti ritatem non solum pro quolibet puncto, quod iam ternis coordinatis definitur n ad quodvis tempus assignari debet, ex quo celeritas quaesita, tanquam Fu tuor variabilium, trium scilicet coordinatarum et temporis, erit spectanda. Quo ulus Integralis generalissime ita definiri poterit, ut dicatur esse methodus ctionem quotcunque variabilium investigandi, cuius Differentialia cuiusque o positam teneant relationem. Quicquid autem adhuc in hoc genere est praestitum um fere casum, quo functio unius variabilis ex data Differentialium relatione quae arum admodum, quod quidem ad functiones plurium variabilium pertineat, in me eometris est allatum. In quo cum quasi Calculi Integralis pars altera sit constitu ri cogimur, eam etiam nunc fere totam incultam iacere. Interim tamen certum versam Theoriam motus fluidorum huic Analyseos parti maximam partem inni no vix quicquam solidi ante expectari posse, quam fines Analyseos etiam in hoc g l mediocriter fuerint prolati. Fortiori certe incitamento Geometris haud erit op es viros ad hoc quasi novum Analyseos genus excolendum intendant.

1. Considero hic aequationes differentiales primi gradus, quae duas ta iabiles involvunt, quas propterea sub hac forma generali

$$Mdx + Ndy = 0$$

raesentare licet, siquidem M et N denotent functiones quascunque bin iabilium x et y[1]). Demonstratum autem est, huiusmodi aequationem se am relationem inter variabiles x et y exprimere, qua pro quovis v us certi valores pro altera definiantur. Cum autem per integrationem tio finita inter ambas variabiles inveniri debeat, aequatio integrali dem ad omnem amplitudinem extendatur, novam quantitatem constan pict, quae, dum penitus ab arbitrio nostro pendet, infinitas quasi ae es integrales complectitur, quae omnes aequationi differentiali a veniant.

1) Confer *Institutiones calculi integralis*, vol. I, § 443—538, ubi magna pars eorum, quae nentatione continentur, invenitur. *Leonhardi Euleri Opera omnia*, series I, vol. 12. H

2. Proposita igitur ...

$$Mdx + Ndy = 0$$

tota vis Analyseos in hoc consistit, ut aequatio finita inter
x et y eliciatur, quae eandem inter illas relationem exprimat,
rentialis, et quidem latissimo sensu, ita ut constantem quam
quae in differentiali non inest, contineat. Verum si haec qu
lissime proponatur, nulla plano adhuc inventa est via ad eiu
veniendi; atque omnes casus, quos adhuc resolvere licuit, ad n
exiguum reduci possunt, ita ut in hac Analyseos parte, pori
maxima adhuc incrementa desiderentur; neque ob hanc caus
omnium huius scientiae arcanorum cognitio expectari queat.

3. Quae quidem adhuc in hoc negotio sunt praestita,
hos casus referri possunt, quibus aequatio differentialis

$$Mdx + Ndy = 0$$

vel sponte separationem variabilium admittit, vel per idon
ad talem formam reduci potest. Quodsi enim introducendis
novis variabilibus v et z, aequatio differentialis proposita in h

$$Vdv + Zdz = 0$$

transmutari queat, in qua V sit functio ipsius v tantum, et
totum negotium erit confectum, dum aequatio integralis e

$$\int Vdv + \int Zdz = \text{Const.},$$

quae manifesto illam constantem arbitrariam per general
invectam complectitur. Atque huc fere redeunt omnia artificia
adhuc in resolutione huiusmodi aequationum sunt usi.

4. Nisi igitur aequatio proposita differentialis sponte se
bilium admittat, totum negotium in hoc consumi est solitum
stitutiones, quae ad separationem viam parent, investigar
saepius summam sagacitatem, quam Geometrae ad scopum
buerunt, admirari oportet. Interim tamen cum nulla certa
modi substitutiones investigandi, haec methodus minus ad r
accommodata, ex quo constitui, aliam methodum non nova
tamen etiamnunc non satis excultam, accuratius perpend

thodum, velut partem, in se complectitur.

5. Aequatione differentiali ad hanc formam

$$Mdx + Ndy = 0$$

ducta, consideretur formula $Mdx + Ndy$ sine respectu habito, quo
nescere debeat, et examinetur, utrum ea sit differentiale cuiuspiam fun
ipsarum x et y, nec ne? Quemadmodum hoc examen sit instituen
passim abunde est explicatum; utramque scilicet functionem M e
erentiari oportet, et cum earum differentialia huiusmodi formam
itura:

$$dM = pdx + qdy \quad \text{et} \quad dN = rdx + sdy,$$

piciatur, utrum sit $q = r$, nec ne? Quodsi enim fuerit $q = r$, hoc infall
criterium, formulam $Mdx + Ndy$ esse integrabilem: at si non f
$= r$, aeque certum est, istam formulam ex nullius finitae functionis ipsa
t y differentiatione esse ortam. Ex quo tota quaestio ad duos casus r
r, quorum alter locum habet, si fuerit $q = r$, alter vero, si hae quanti
t r non fuerint inter se aequales.

6. Ad aequalitatem igitur, vel inaequalitatem, quantitatum q et r ag
dam, ne opus quidem est, ut functiones M et N penitus per differe
nem evolvantur, sed sufficit in functione M, quae cum dx est coniun
ntitatem x ut constantem spectare, eamque tantum eius differen
tem quaerere, quae ex variabilitate ipsius y tantum nascitur, si qui
modo membrum qdy obtinetur, valorem autem ipsius q sic erutum
ptione $\left(\frac{dM}{dy}\right)$ denotare soleo. Simili modo altera functio N, quae cu
coniuncta, ita differentietur, ut y pro constante tractetur, et ex v
tate solius x impetretur differentialis pars rdx, ubi valorem ipsius r pa
$\left(\frac{dN}{dx}\right)$ exprimo. Quodsi ergo formula $Mdx + Ndy$ ita fuerit compa
sit

$$\left(\frac{dM}{dy}\right) = \left(\frac{dN}{dx}\right),$$

est integrabilis, eiusque integrale sequenti modo inveniri poterit.
o, si hoc criterium non locum habeat, videamus quomodo sit procedend

$$Mdx + Ndy = 0$$

ita fuerit comparata, ut sit

$$\left(\frac{dM}{dy}\right) = \left(\frac{dN}{dx}\right),$$

invenire eius aequationem integralem.

SOLUTIO

Si fuerit

$$\left(\frac{dM}{dy}\right) = \left(\frac{dN}{dx}\right),$$

tunc datur functio finita binarum variabilium x et y, quae differ
$Mdx + Ndy$. Sit V ista functio, et cum sit

$$dV = Mdx + Ndy,$$

erit Mdx differentiale ipsius V, si tantum x variabile sumatu
differentiale, si tantum y variabile capiatur. Hinc ergo vicissi
si vel Mdx integretur, spectata y ut constante, vel Ndy integr
ut constante: sicque haec operatio reducitur ad integrati
differentialis unicam variabilem involventis, quae in hoc neg
braice succedat, sive quadraturas curvarum requirat, concedi
autem hac ratione quantitas V duplici modo inveniatur, et a
vice constantis functionem quamcunque ipsius y, altera vero ip
ita ut sit

$$\text{et } V = \int Mdx + Y \text{ et } V = \int Ndy + X,$$

semper has functiones Y ipsius y et X ipsius x ita definir
$\int Mdx + Y = \int Ndy + X$, id quod quovis casu facile praesta
cum quantitas V sit integrale formulae $Mdx + Ndy$, evidens
propositae $Mdx + Ndy = 0$ integralem aequationem fore $V =$
completam, propterea quod involvit constantem quantitat
nostro pendentem.

COROLLARIUM I

8. In hoc problemate statim continetur casus aequationu
Si enim fuerit M functio ipsius x tantum, et N functio ipsius
utique

$$\left(\frac{dM}{dy}\right) = 0 \quad \text{et} \quad \left(\frac{dN}{dx}\right) = 0, \quad \text{ideoque} \quad \left(\frac{dM}{dy}\right) = \left(\frac{dN}{dx}\right);$$

rgo casus simplicissimus, quem problema in se complectitur.

COROLLARIUM 2

nodsi autem in aequatione differentiali

$$Mdx + Ndy = 0$$

functio solius x, et N solius y, utraque pars seorsim integrabilis que aequatio integralis erit:

$$\int Mdx + \int Ndy = \text{Const.}$$

COROLLARIUM 3

Praeterea vero nostrum problema resolutionem infinitarum aliarum um differentialium largitur, quarum omnium character communis nsistit, ut sit

$$\left(\frac{dM}{dy}\right) = \left(\frac{dN}{dx}\right),$$

resolutio per integrationem formularum, unicam variabilem conti- expediri potest.

SCHOLION 1

Quoties ergo in aequatione differentiali $Mdx + Ndy = 0$ fuerit $\left.\frac{N}{dx}\right)$, eius resolutio nullam habet difficultatem, dummodo integratio m unicam variabilem involventium concedatur; quam quidem iure licet. Interim tamen determinatio functionum illarum X et Y, quae antium introduci debent, molestiam quandam creare videri posset, m singulis casibus mox evanescere reperietur. Verum quo magis peratio contrahatur, ne duplici quidem integratione est opus. Post- n altera pars Mdx, spectata y tanquam constanti, fuerit integrata, grale sit $= Q$, statuatur

$$V = Q + Y,$$

tisper Y pro functione indefinita ipsius y, in quam altera variabilis x on ingrediatur. Tum differentietur denuo haec quantitas $Q + Y$, x tanquam constantem, et quia differentiale prodire debet $= Ndy$,

$$Mdx + Ndy = 0$$

ita fuerit comparata, ut sit

$$\left(\frac{dM}{dy}\right) = \left(\frac{dN}{dx}\right),$$

invenire eius aequationem integralem.

SOLUTIO

Si fuerit

$$\left(\frac{dM}{dy}\right) = \left(\frac{dN}{dx}\right),$$

tunc datur functio finita binarum variabilium x et
$Mdx + Ndy$. Sit V ista functio, et cum sit

$$dV = Mdx + Ndy,$$

erit Mdx differentiale ipsius V, si tantum x va
differentiale, si tantum y variabile capiatur. Hin
si vel Mdx integretur, spectata y ut constante, v
ut constante: sicque haec operatio reducitur
differentialis unicam variabilem involventis, qu
braice succedat, sive quadraturas curvarum requ
autem hac ratione quantitas V duplici modo in
vice constantis functionem quamcunque ipsius y,
ita ut sit

$$\text{et} \quad V = \int Mdx + Y \quad \text{et} \quad V =$$

semper has functiones Y ipsius y et X ipsius
$\int Mdx + Y = \int Ndy + X$, id quod quovis casu
cum quantitas V sit integrale formulae $Mdx +$
propositae $Mdx + Ndy = 0$ integralem aequatio
completam, propterea quod involvit constant
nostro pendentem.

COROLLARIUM 1

8. In hoc problemate statim continetur casu
Si enim fuerit M functio ipsius x tantum, et N
utique

$$Mdx + Ndy = 0$$

erit M functio solius x, et N solius y, utraque pars seorsim integrabil
istit, atque aequatio integralis erit:

$$\int Mdx + \int Ndy = \text{Const.}$$

COROLLARIUM 3

10. Praeterea vero nostrum problema resolutionem infinitarum aliaru
quationum differentialium largitur, quarum omnium character commun
hoc consistit, ut sit

$$\left(\frac{dM}{dy}\right) = \left(\frac{dN}{dx}\right),$$

rumque resolutio per integrationem formularum, unicam variabilem cont
ntium, expediri potest.

SCHOLION 1

11. Quoties ergo in aequatione differentiali $Mdx + Ndy = 0$ fuer
$\left(\frac{dM}{dy}\right) = \left(\frac{dN}{dx}\right)$, eius resolutio nullam habet difficultatem, dummodo integrati
rmularum unicam variabilem involventium concedatur; quam quidem im
stulare licet. Interim tamen determinatio functionum illarum X et Y, qua
co constantium introduci debent, molestiam quandam creare videri posse
ae autem singulis casibus mox evanescere reperietur. Verum quo mag
haec operatio contrahatur, ne duplici quidem integratione est opus. Pos
am enim altera pars Mdx, spectata y tanquam constanti, fuerit integrat
od integrale sit $= Q$, statuatur

$$V = Q + Y,$$

sito tantisper Y pro functione indefinita ipsius y, in quam altera variabilis
orsus non ingrediatur. Tum differentietur denuo haec quantitas $Q + Y$
ctando x tanquam constantem, et quia differentiale prodire debet $= Nd$

7. Si aequatio differentialis

$$Mdx + Ndy = 0$$

ita fuerit comparata, ut sit

$$\left(\frac{dM}{dy}\right) = \left(\frac{dN}{dx}\right),$$

invenire eius aequationem integralem.

SOLUTIO

Si fuerit

$$\left(\frac{dM}{dy}\right) = \left(\frac{dN}{dx}\right),$$

tunc datur functio finita binarum variabilium x et y, quae d
$Mdx + Ndy$. Sit V ista functio, et cum sit

$$dV = Mdx + Ndy,$$

erit Mdx differentiale ipsius V, si tantum x variabile sun
differentiale, si tantum y variabile capiatur. Hinc ergo vic
si vel Mdx integretur, spectata y ut constante, vel Ndy int
ut constante: sicque haec operatio reducitur ad integr
differentialis unicam variabilem involventis, quae in hoc
braice succedat, sive quadraturas curvarum requirat, concec
autem hac ratione quantitas V duplici modo inveniatur, e
vice constantis functionem quamcunque ipsius y, altera vero
ita ut sit

$$\text{et} \quad V = \int Mdx + Y \quad \text{et} \quad V = \int Ndy + X$$

semper has functiones Y ipsius y et X ipsius x ita defin
$\int Mdx + Y = \int Ndy + X$, id quod quovis casu facile prac
cum quantitas V sit integrale formulae $Mdx + Ndy$, evider
propositae $Mdx + Ndy = 0$ integralem aequationem fore V
completam, propterea quod involvit constantem quantita
nostro pendentem.

COROLLARIUM 1

8. In hoc problemate statim continetur casus aequation
Si enim fuerit M functio ipsius x tantum, et N functio ipsi
utique

$$\left(\frac{dM}{dy}\right) = 0 \quad \text{et} \quad \left(\frac{dN}{dx}\right) = 0, \quad \text{ideoque} \quad \left(\frac{dM}{dy}\right) = \left(\frac{dN}{dx}\right);$$

ui est ergo casus simplicissimus, quem problema in se complectitur.

COROLLARIUM 2

9. Quodsi autem in aequatione differentiali

$$Mdx + Ndy = 0$$

uerit M functio solius x, et N solius y, utraque pars seorsim integra sistit, atque aequatio integralis erit:

$$\int Mdx + \int Ndy = \text{Const.}$$

COROLLARIUM 3

10. Praeterea vero nostrum problema resolutionem infinitarum alia equationum differentialium largitur, quarum omnium character comm n hac consistit, ut sit

$$\left(\frac{dM}{dy}\right) = \left(\frac{dN}{dx}\right),$$

arumque resolutio per integrationem formularum, unicam variabilem co entium, expediri potest.

SCHOLION 1

11. Quoties ergo in aequatione differentiali $Mdx + Ndy = 0$ fu $\left(\frac{dM}{dy}\right) = \left(\frac{dN}{dx}\right)$, eius resolutio nullam habet difficultatem, dummodo integr ormularum unicam variabilem involventium concedatur; quam quidem ostulare licet. Interim tamen determinatio functionum illarum X et Y, q co constantium introduci debent, molestiam quandam creare videri pos uae autem singulis casibus mox evanescere reperietur. Verum quo m haec operatio contrahatur, ne duplici quidem integratione est opus. P uam enim altera pars Mdx, spectata y tanquam constanti, fuerit integr uod integrale sit $= Q$, statuatur

$$V = Q + Y,$$

osito tantisper Y pro functione indefinita ipsius y, in quam altera variabi rorsus non ingrediatur. Tum differentietur denuo haec quantitas $Q +$ actando x tanquam constantem, et quia differentiale prodire debet $= N$

integralis erit $Q + Y = \text{Const.}$, quam operationem sequentibus e
strari conveniet.

EXEMPLUM 1

12. *Integrare hanc aequationem differentialem:*

$$2axydx + axxdy - y^3dx - 3xyydy = 0.$$

Comparata hac aequatione cum forma $Mdx + Ndy = 0$, erit:

$$M = 2axy - y^3 \text{ et } N = axx - 3xyy.$$

Primum igitur dispiciendum est, utrum hic casus in problemate
quem in finem quaeramus valores:

$$\left(\frac{dM}{dy}\right) = 2ax - 3yy \text{ et } \left(\frac{dN}{dx}\right) = 2ax - 3yy,$$

qui cum sint aequales, operatio praescripta necessario succedet.
autem, sumta y pro constante:

$$\int Mdx = axxy - y^3x + Y;$$

cuius formae si differentiale sumatur, posita x constante, prodil

$$axxdy - 3yyxdy + dY = Ndy,$$

et pro N valore suo $axx - 3xyy$ restituto, fiet $dY = 0$, ex quo nas
vel $Y = \text{const.}$ Quare aequatio integralis quaesita habebitur:

$$axxy - y^3x = \text{Const.}$$

EXEMPLUM 2

13. *Integrare hanc aequationem differentialem:*

$$\frac{ydy + xdx - 2ydx}{(y-x)^2} = 0.$$

Comparata hac aequatione cum forma $Mdx + Ndy = 0$, erit:

$$M = \frac{x - 2y}{(y-x)^2} \text{ et } N = \frac{y}{(y-x)^2}.$$

$$\left(\frac{dM}{dy}\right)=\frac{2y}{(y-x)^3} \quad \text{et} \quad \left(\frac{dN}{dx}\right)=\frac{2y}{(y-x)^3},$$

i cum sint aequales, negotium succedet. Quare secundum regulam tur, sumto y constante, integrale:

$$\int Mdx=\int\frac{xdx-2ydx}{(y-x)^2}=-\int\frac{dx}{y-x}-\int\frac{ydx}{(y-x)^2}$$

reperietur:

$$\int Mdx=l(y-x)-\frac{y}{y-x}+Y,$$

ius differentiale, sumto x constante, producere debet alteram aequa opositae partem Ndy; unde habebitur:

$$Ndy=\frac{dy}{y-x}+\frac{xdy}{(y-x)^2}+dY=\frac{ydy}{(y-x)^2}+dY.$$

m igitur sit

$$Ndy=\frac{ydy}{(y-x)^2}, \quad \text{erit} \quad dY=0 \quad \text{et} \quad Y=0,$$

nstantem enim in Y negligere licet, quia iam in aequationem integ troducitur, quippe quae erit:

$$l(y-x)-\frac{y}{y-x}=\text{Const.}$$

EXEMPLUM 3

14. *Integrare hanc aequationem differentialem:*

$$\frac{dx}{x}+\frac{yydx}{x^3}-\frac{ydy}{xx}+\frac{(ydx-xdy)\sqrt{(xx+yy)}}{x^3}=0.$$

omparata hac aequatione cum forma $Mdx+Ndy=0$, habebimus:

$$M=\frac{xx+yy+y\sqrt{(xx+yy)}}{x^3} \quad \text{et} \quad N=\frac{-y-\sqrt{(xx+yy)}}{xx},$$

nde pro criterio explorando quaeratur:

$$\left(\frac{dM}{dy}\right)=\frac{2y}{x^3}+\frac{\sqrt{(xx+yy)}}{x^3}+\frac{yy}{x^3\sqrt{(xx+yy)}}$$

;

$$\left(\frac{dN}{dx}\right) = \frac{\ldots}{x^3} \ldots \frac{\ldots}{x^3} \ldots \frac{\ldots}{xx\sqrt{(xx\ldots}}$$

qui valores reducti cum fiant aequales, scilicet

$$\left(\frac{dM}{dy}\right) = \left(\frac{dN}{dx}\right) = \frac{2y}{x^3} + \frac{xx+2yy}{x^3\sqrt{(xx+y\ldots}}$$

resolutio erit in potestate. Investigetur ergo, sumto

$$\int Mdx = lx - \frac{yy}{2xx} + y\int\frac{dx}{x^3}\sqrt{(xx\ldots}$$

At per regulas integrandi formulas unicam variabilem
pro constante habetur, reperitur:

$$\int\frac{ydx}{x^3}\sqrt{(xx+yy)} = -\frac{y\sqrt{(xx+yy)}}{2xx} + \frac{1}{2}l\ldots$$

ita ut sit:

$$\int Mdx = lx - \frac{yy}{2xx} - \frac{y\sqrt{(xx+yy)}}{2xx} + \frac{1}{2}l\frac{\sqrt{(xx\ldots}}{}$$

At huius quantitatis differentiale, assumto x pro const

$$Ndy = \frac{-ydy-dy\sqrt{(xx+yy)}}{xx}\ldots$$

nanciscemur:

$$Ndy = \frac{-ydy}{xx} - \frac{dy\sqrt{(xx+yy)}}{2xx} - \frac{yydy}{2xx\sqrt{(xx+yy)}} \ldots \frac{dy}{2y\ldots}$$

qua forma cum illa comparata fit:

$$dY = \frac{dy\sqrt{(xx+yy)}}{2xx} + \frac{yydy}{2xx\sqrt{(xx+yy)}} + \frac{dy}{2y\ldots}$$

ubi termini, qui adhuc continent x, sponte se destru

$$dY = \frac{dy}{2y} \quad\text{et}\quad Y = \frac{1}{2}ly.$$

Quo valore pro Y invento, obtinebitur aequatio int

$$lx - \frac{yy}{2xx} - \frac{y\sqrt{(xx+yy)}}{2xx} + \frac{1}{2}l(1+\sqrt{(xx+yy)}\ldots$$

... Ex his exemplis satis perspicitur, quemadmodum perpetuo ope... praescripta sit instituenda, ita ut hinc nulla amplius difficultas moles... cessat, nisi quae ex integratione formularum unicam variabilem invol... sui quandoque relinquitur, dum integratio neque algebraice absolvi, n... circuli hyperbolaeve quadraturam reduci patitur. Verum tum super... quadraturas simili modo tractari oportet, et si quae difficultates relinquan... o non huic methodo sunt adscribendae. Quam ob rem hic assumere ... otius aequatio differentialis

$$Mdx + Ndy = 0$$

... fuerit comparata, ut in ea sit

$$\left(\frac{dM}{dy}\right) = \left(\frac{dN}{dx}\right),$$

...ties integrationem esse in nostra potestate; unde ad eas aequationes p... quibus hoc criterium non habet locum.

THEOREMA

16. Si in aequatione differentiali

$$Mdx + Ndy = 0$$

...n fuerit

$$\left(\frac{dM}{dy}\right) = \left(\frac{dN}{dx}\right),$$

...mper datur multiplicator, per quem formula $Mdx + Ndy$ multiplicata ...tegrabilis[1]).

DEMONSTRATIO

Cum non sit

$$\left(\frac{dM}{dy}\right) = \left(\frac{dN}{dx}\right),$$

...tam formula $Mdx + Ndy$ non erit integrabilis, seu nulla existit functio ...m x et y, cuius differentiale sit $Mdx + Ndy$. Verum hic non tam form... $dx + Ndy$, quam aequationis $Mdx + Ndy = 0$, quaeritur integrale...

1) Revera Eulerus hoc ibi non ostendit. Cf. § 48 necnon *Institutiones calculi inte...* ... I, § 459. Vide notam p. 337.

cum eadem aequatio subsistat, si per functionem quam...
et y multiplicetur, ita ut sit

$$LMdx + LNdy = 0,$$

demonstrandum est, semper eiusmodi dari functionem ...

$$LMdx + LNdy$$

fiat integrabilis. Quo enim hoc eveniat, necesse est, ut s...

$$\left(\frac{d \cdot LM}{dy}\right) = \left(\frac{d \cdot LN}{dx}\right),$$

vel si ponatur

$$dL = Pdx + Qdy,$$

cum sit

$$\left(\frac{dL}{dy}\right) = Q \text{ et } \left(\frac{dL}{dx}\right) = P,$$

functio L ita debet esse comparata, ut sit:

$$L\left(\frac{dM}{dy}\right) + MQ = L\left(\frac{dN}{dx}\right) + NP.$$

Evidens autem est, hanc conditionem sufficere ad defini...
per quam si formula $Mdx + Ndy$ multiplicetur, fiat in...

COROLLARIUM 1

17. Invento ergo tali multiplicatore L, qui reddat

$$Mdx + Ndy$$

integrabilem, aequatio $Mdx + Ndy = 0$, in formam

$$LMdx + LNdy = 0$$

translata, integrari poterit methodo in problemate prae...

COROLLARIUM 2

18. Quaeratur scilicet, spectata y tanquam constant...
ad quod adiiciatur talis functio Y ipsius y, ut, si aggre...

$$\int LMdx + Y$$

denuo differentietur, spectata iam x ut constante, prode...
erit aequatio integralis

$$\int LMdx + Y = \text{Const.}$$

... multiplicator ignur L ita debet esse comparatus, ut posito

$$dL = Pdx + Qdy,$$

... iat huic aequationi:

$$L\left(\frac{dM}{dy}\right) + MQ = L\left(\frac{dN}{dx}\right) + NP$$

... nic:

$$\frac{NP - MQ}{L} = \left(\frac{dM}{dy}\right) - \left(\frac{dN}{dx}\right),$$

... manifestum est, si esset

$$\left(\frac{dM}{dy}\right) = \left(\frac{dN}{dx}\right),$$

... sumi posse unitatem, vel quantitatem constantem quamcunque, ... $= 0$ et $Q = 0$.

SCHOLION

...20. Si ergo hinc in genere multiplicator L inveniri posset, haberetur ... lis resolutio omnium aequationum differentialium primi gradus; id ... erare quidem licet. Contentos ergo nos esse oportet, si pro variis casi... busque aequationum differentialium generibus, huiusmodi factores ... garo valeamus. Sunt autem duo aequationum genera, pro quibus ... res commode erui possunt, quorum alterum eas comprehendit aequatio... ibus altera variabilis nusquam ultra unam dimensionem exsurgit; alte... genus est aequationum homogenearum. Praeter haec vero duo ge... s alii existunt casus, quibus inventio talis factoris absolvi potest, ... ntius examinasse, usu non carebit, cum haec sola via patere vide... m Analyseos partem, quae adhuc desideratur, excolendam ac perfi... Quam ob rem hic constitui, plura aequationum genera colligere, ... uiusmodi multiplicatorem ad integrabilitatem perduci possunt.

PROBLEMA 2

...21. Cognito uno multiplicatore L, qui formulam $Mdx + Ndy$ inte... reddit, invenire infinitos alios multiplicatores, qui idem officium praes...

SOLUTIO

Cum formula $L(Mdx + Ndy)$ per hypothesin sit integrabilis, sit ... ralo $= z$, ita ut sit

existente z quapiam functione ipsarum x et y. Denotet iam
quamcunque ipsius z, et quia formula Zdz est etiam integrabil

$$Zdz = LZ\,(Mdx + Ndy),$$

manifestum est formulam propositam $Mdx + Ndy$ quoque fier
si per LZ multiplicetur. Dato ergo uno multiplicatore L,
$Mdx + Ndy$ integrabilem reddat, ex eo innumerabiles alii facto
possunt, qui idem sint praestituri, sumendo pro Z functione
integralis

$$\int L\,(Mdx + Ndy).$$

COROLLARIUM 1

22. Proposita igitur formula differentiali quacunque Md
solum unus, sed etiam infiniti dantur multiplicatores, qui ea
reddant. Quorum autem unum invenisse sufficit, cum reliqui o
determinentur.

COROLLARIUM 2

23. Si ergo habeatur aequatio differentialis

$$Mdx + Ndy = 0,$$

ea infinitis modis ad integrabilitatem perduci potest. Sive a
multiplicator L, sive alius quicunque LZ, aequatio integralis
redit; siquidem ille factor L praebet $z =$ Const., hic vero $\int Z$
quod convenit, cum $\int Zdz$ sit functio ipsius z.

EXEMPLUM 1

24. *Invenire omnes multiplicatores, qui reddant hanc form*

$$\alpha ydx + \beta xdy$$

integrabilem.

Unus multiplicator hoc praestans in promtu est, scilicet
$L = \frac{1}{xy}$, fiatque

$$dz = \frac{\alpha ydx + \beta xdy}{xy} = \frac{\alpha dx}{x} + \frac{\beta dy}{y},$$

notet iam Z functionem quamcunque ipsius $z = lx^\alpha y^\beta$, hoc est ipsius x
que omnes multiplicatores quaesiti in hac forma generali

$$\frac{1}{xy} \text{ funct. } x^\alpha y^\beta$$

ntinebuntur.

Simpliciores ergo multiplicatores reperientur, si loco functionis pote
uaecunque ipsius $x^\alpha y^\beta$ capiatur; sicque formula $\alpha y dx + \beta x dy$ integra
dditur per hunc multiplicatorem latius patentem $x^{\alpha n-1} y^{\beta n-1}$. Si magis c
siti desiderentur, plures huiusmodi utcunque inter se combinari poter
habeantur

$$A x^{\alpha n-1} y^{\beta n-1} + B x^{\alpha m-1} y^{\beta m-1} + \text{etc.}$$

EXEMPLUM 2

25. *Invenire omnes multiplicatores, qui reddant hanc formulam diff*
lem

$$\alpha x^{\mu-1} y^\nu dx + \beta x^\mu y^{\nu-1} dy$$

egrabilem.

Hic iterum statim se offert unus multiplicator

$$L = \frac{1}{x^\mu y^\nu},$$

i praebet

$$dz = \frac{\alpha dx}{x} + \frac{\beta dy}{y},$$

de fit

$$z = \alpha lx + \beta ly = lx^\alpha y^\beta.$$

sito igitur Z pro functione quacunque ipsius $x^\alpha y^\beta$, omnes multiplica
ntinebuntur in hac expressione generali

$$\frac{Z}{x^\mu y^\nu} = \frac{1}{x^\mu y^\nu} \text{ funct. } x^\alpha y^\beta.$$

loco istius functionis sumatur potestas quaecunque $x^{\alpha n} y^{\beta n}$, innumeri
tinebuntur multiplicatores, unico termino constantes $x^{\alpha n-\mu} y^{\beta n-\nu}$, sum
o n numeros quoscunque.

$$\alpha x^{\mu-1} y^{\nu} dx + \beta x^{\mu} y^{\nu-1} dy$$

communem recipiant multiplicatorem: quod si eveniat, aequati
ex huiusmodi formulis, tanquam membris, composita integrabilis
dum multiplicator iste communis adhibetur. Quem casum iam c
evolvamus.

PROBLEMA 3

27. Proposita sit ista aequatio differentialis:

$$\alpha y dx + \beta x dy + \gamma x^{\mu-1} y^{\nu} dx + \delta x^{\mu} y^{\nu-1} dy = 0,$$

cuius integralem inveniri oporteat.

SOLUTIO

Ad multiplicatorem idoneum inveniendum, quo haec aeq
integrabilis, consideretur utrumque membrum seorsim. Ac
membrum $\alpha y dx + \beta x dy$ vidimus integrabile reddi hoc multip

$$x^{\alpha n-1} y^{\beta n-1},$$

posterius vero membrum $\gamma x^{\mu-1} y^{\nu} dx + \delta x^{\mu} y^{\nu-1} dy$ hoc

$$x^{\gamma m-\mu} y^{\delta m-\nu}.$$

Quia nunc pro n et m numeros quoscunque accipere licet, hi d
aequalitatem reduci poterunt; unde fit

$$\alpha n - 1 = \gamma m - \mu \quad \text{et} \quad \beta n - 1 = \delta m - \nu$$

ideoque

$$n = \frac{\gamma m - \mu + 1}{\alpha} = \frac{\delta m - \nu + 1}{\beta},$$

hincque obtinetur

$$m = \frac{\alpha\nu - \beta\mu - \alpha + \beta}{\alpha\delta - \beta\gamma} \quad \text{et} \quad n = \frac{\gamma\nu - \delta\mu - \gamma + \delta}{\alpha\delta - \beta\gamma}.$$

His valoribus pro m et n inventis, iste multiplicator commu
aequationem integralem:

$$\frac{1}{n} x^{\alpha n} y^{\beta n} + \frac{1}{m} x^{\gamma m} y^{\delta m} = \text{Const.}$$

28. Haec ergo aequatio integralis semper est algebraica, siquidem pro m et n valores veri reperiantur. Ii igitur tantum casus singulari reductione egent, quibus numeri m et n vel in infinitum abeunt, vel evanescunt.

COROLLARIUM 2

29. Infiniti autem evadunt ambo numeri m et n, si fuerit $\alpha\delta = \beta\gamma$. Verum hoc casu ipsa aequatio differentialis in duos factores resolvitur, hancque formam acquirit

$$(\alpha y\,dx + \beta x\,dy)\left(1 + \frac{\gamma}{\alpha}x^{\mu-1}y^{\nu-1}\right) = 0$$

eritque erit

$$\text{vel}\ \ \alpha y dx + \beta x dy = 0, \ \ \text{vel}\ \ 1 + \frac{\gamma}{\alpha}x^{\mu-1}y^{\nu-1} = 0,$$

quarum resolutionum neutra difficultate laborat.

COROLLARIUM 3

30. At si fiat $n = 0$, seu

$$\gamma(\nu - 1) = \delta(\mu - 1),$$

consideretur numerus n ut valde parvus, et cum sit per seriem convergentem $x^{\alpha n} = 1 + \alpha n lx + \frac{1}{2}\alpha^2 n^2 (lx)^2 + \text{etc. et } y^{\beta n} = 1 + \beta n ly + \frac{1}{2}\beta^2 n^2 (ly)^2 + \text{etc.},$

erit

$$\frac{1}{n}x^{\alpha n}y^{\beta n} = \frac{1}{n} + \alpha lx + \beta ly = lx^\alpha y^\beta$$

prima parte $\frac{1}{n}$ in constantem involuta. Hoc ergo casu erit aequatio integralis

$$lx^\alpha y^\beta + \frac{1}{m}x^{\gamma m}y^{\delta m} = \text{Const.}$$

COROLLARIUM 4

31. Statuatur ergo pro hoc casu

$$\mu = \gamma k + 1 \ \ \text{et} \ \ \nu = \delta k + 1,$$

ut habeatur ista aequatio differentialis:

$$m = \frac{\alpha\delta k - \beta\gamma k}{\alpha\delta - \beta\gamma} = k,$$

erit aequatio integralis

$$lx^\alpha y^\beta + \frac{1}{k}x^{\gamma k}y^{\delta k} = \text{Const.}$$

COROLLARIUM 5

32. Simili modo si fuerit $m = 0$, seu

$$\alpha(\nu - 1) = \beta(\mu - 1),$$

ob

$$\frac{1}{m}x^{\gamma m}y^{\delta m} = lx^\gamma y^\delta,$$

si ponatur $\mu = \alpha k + 1$ et $\nu = \beta k + 1$, unde fit

$$n = \frac{\gamma\beta k - \delta\alpha k}{\alpha\delta - \beta\gamma} = -k,$$

erit huius aequationis

$$\alpha y dx + \beta x dy + \gamma x^{\alpha k}y^{\beta k + 1}dx + \delta x^{\alpha k + 1}y^{\beta k}dy = 0$$

integrale

$$-\frac{1}{k}x^{-\alpha k}y^{-\beta k} + lx^\gamma y^\delta = \text{Const.}$$

SCHOLION

33. Neque vero huiusmodi resolutio in membra, quae per eundo
plicatorem reddantur integrabilia, ad omnis generis aequationes patet
enim utique potest, ut tota aequatio per quampiam quantitatem mu
integrabilis evadat, cum tamen nulla eius pars inde seorsim integrabili
ex quo huic tractationi, qua hic sum usus, non nimis tribui oportet.

PROBLEMA 4

34. Si proposita sit aequatio differentialis

$$Pdx + Qydx + Rdy = 0,$$

ubi P, Q et R denotant functiones quascunque ipsius x, ita ut altera

plus una dimensione non habeat, invenire multiplicatorem, qui eam reddat integrabilem.

SOLUTIO

Comparata hac aequatione cum forma $Mdx + Ndy = 0$ erit

$$M = P + Qy \text{ et } N = R,$$

unde fiet

$$\left(\frac{dM}{dy}\right) = Q \quad \text{et} \quad \left(\frac{dN}{dx}\right) = \frac{dR}{dx}.$$

Statuatur iam L pro multiplicatore quaesito, sitque

$$dL = pdx + qdy,$$

atque huic aequationi satisfieri oportet

$$\frac{Np - Mq}{L} = Q - \frac{dR}{dx} = \frac{Rp - (P + Qy)q}{L}.$$

Cum iam sit $Q - \frac{dR}{dx}$ functio ipsius x tantum, pro L quoque functio ipsius x tantum accipi poterit, ita ut sit $q = 0$, et $dL = pdx$; unde erit:

$$Q - \frac{dR}{dx} = \frac{Rp}{L}, \quad \text{seu} \quad Qdx - dR = \frac{RdL}{L}$$

ideoque

$$\frac{dL}{L} = \frac{Qdx}{R} - \frac{dR}{R}.$$

Quare integrando habebitur

$$lL = \int \frac{Qdx}{R} - lR,$$

et sumto e pro numero, cuius logarithmus hyperbolicus est unitas, prodit

$$L = \frac{1}{R} e^{\int \frac{Qdx}{R}}.$$

Invento autem hoc multiplicatore erit aequatio integralis:

$$\int \frac{Pdx}{R} e^{\int \frac{Qdx}{R}} + ye^{\int \frac{Qdx}{R}} = \text{Const.}$$

$$m = \frac{\quad}{\alpha\delta - \beta\gamma} = k,$$

erit aequatio integralis

$$lx^{\alpha}y^{\beta} + \frac{1}{k}x^{\gamma k}y^{\delta k} = \text{Const.}$$

COROLLARIUM 5

32. Simili modo si fuerit $m = 0$, seu

$$\alpha(\nu - 1) = \beta(\mu - 1),$$

ob

$$\frac{1}{m}x^{\gamma m}y^{\delta m} = lx^{\gamma}y^{\delta},$$

si ponatur $\mu = \alpha k + 1$ et $\nu = \beta k + 1$, unde fit

$$n = \frac{\gamma\beta k - \delta\alpha k}{\alpha\delta - \beta\gamma} = -k,$$

erit huius aequationis

$$\alpha y dx + \beta x dy + \gamma x^{\alpha k} y^{\beta k+1} dx + \delta x^{\alpha k}$$

integrale

$$-\frac{1}{k}x^{-\alpha k}y^{-\beta k} + lx^{\gamma}y^{\delta} = \text{Con}$$

SCHOLION

33. Neque vero huiusmodi resolutio in membra plicatorem reddantur integrabilia, ad omnis generis enim utique potest, ut tota aequatio per quampiam integrabilis evadat, cum tamen nulla eius pars inde s ex quo huic tractationi, qua hic sum usus, non nim

PROBLEMA 4

34. Si proposita sit aequatio differentialis

$$Pdx + Qydx + Rdy = 0$$

ubi P, Q et R denotant functiones quascunque ipsiu

parata hac aequatione cum forma $Mdx + Ndy = 0$ erit

$$M = P + Qy \text{ et } N = R,$$

t.

$$\left(\frac{dM}{dy}\right) = Q \text{ et } \left(\frac{dN}{dx}\right) = \frac{dR}{dx}.$$

ur iam L pro multiplicatore quaesito, sitque

$$dL = pdx + qdy,$$

uic aequationi satisfieri oportet

$$\frac{Np - Mq}{L} = Q - \frac{dR}{dx} = \frac{Rp - (P + Qy)q}{L}.$$

m sit $Q - \frac{dR}{dx}$ functio ipsius x tantum, pro L quoque functio ipsius x
a accipi poterit, ita ut sit $q = 0$, et $dL = pdx$; unde erit:

$$Q - \frac{dR}{dx} = \frac{Rp}{L}, \text{ seu } Qdx - dR = \frac{RdL}{L}$$

us

$$\frac{dL}{L} = \frac{Qdx}{R} - \frac{dR}{R}.$$

e integrando habebitur

$$lL = \int \frac{Qdx}{R} - lR,$$

nto e pro numero, cuius logarithmus hyperbolicus est unitas, prodit

$$L = \frac{1}{R} e^{\int \frac{Qdx}{R}}.$$

nto autem hoc multiplicatore erit aequatio integralis:

$$\int \frac{Pdx}{R} e^{\int \frac{Qdx}{R}} + y e^{\int \frac{Qdx}{R}} = \text{Const.}$$

35. Si aequatio habeat formam propositam, ea, ante
tractetur, dividi poterit per R, ut hanc formam induat

$$Pdx + Qydx + dy = 0,$$

seu statim assumere licet $R = 1$, quo facto multiplicator erit
integralis

$$\int e^{\int Qdx} Pdx + e^{\int Qdx} y = \text{Const.}$$

COROLLARIUM 2

36. Si ponatur hoc integrale

$$\int e^{\int Qdx} Pdx + e^{\int Qdx} y = z,$$

ita ut z sit functio quaepiam ambarum variabilium, tum vero Z
nem quamcunque ipsius z: omnes multiplicatores, qui formu

$$Pdx + Qydx + dy$$

reddunt integrabilem, in hac forma generali $e^{\int Qdx} Z$ continen

PROBLEMA 5

37. Si proposita sit aequatio differentialis:

$$Py^n dx + Qydx + Rdy = 0,$$

ubi P, Q et R denotent functiones quascunque ipsius x, inv
torem, qui eam reddat integrabilem.

SOLUTIO

Erit ergo $M = Py^n + Qy$ et $N = R$, hincque

$$\left(\frac{dM}{dy}\right) = nPy^{n-1} + Q \quad \text{et} \quad \left(\frac{dN}{dx}\right) = \frac{dR}{dx}.$$

Quare posito multiplicatore quaesito L et

$$dL = pdx + qdy,$$

erit ex ante inventis:

$$\frac{Rp - Py^n q - Qyq}{L} = nPy^{n-1} + Q - \frac{dR}{dx}.$$

ur $L = Sy^m$, existente S functione ipsius x tantum, erit

$$p = \frac{y^m dS}{dx} \quad \text{et} \quad q = mSy^{m-1},$$

s valoribus substitutis, prodibit:

$$\frac{RdS}{Sdx} - mPy^{n-1} - mQ = nPy^{n-1} + Q - \frac{dR}{dx}.$$

aequatio ut subsistere possit, sumi debet $m = -n$, ac fiet

$$\frac{RdS}{Sdx} = (1-n)Q - \frac{dR}{dx}, \text{ seu } \frac{dS}{S} = \frac{(1-n)Qdx}{R} - \frac{dR}{R}.$$

e cum integrando proveniat

$$S = \frac{1}{R} e^{(1-n)\int \frac{Qdx}{R}},$$

ob $m = -n$, multiplicator quaesitus:

$$L = \frac{y^{-n}}{R} e^{(1-n)\int \frac{Qdx}{R}}$$

equatio integralis erit

$$\frac{y^{1-n}}{1-n} e^{(1-n)\int \frac{Qdx}{R}} + \int \frac{Pdx}{R} e^{(1-n)\int \frac{Qdx}{R}} = \text{Const.}$$

COROLLARIUM 1

38. Si $n = 0$, habemus casum ante tractatum aequationis

$$Pdx + Qydx + Rdy = 0,$$

ae per multiplicatorem

$$\frac{1}{R} e^{\int \frac{Qdx}{R}}$$

tegrabilis redditur; et cuius aequatio integralis est

$$y e^{\int \frac{Qdx}{R}} + \int \frac{Pdx}{R} e^{\int \frac{Qdx}{R}} = \text{Const.}$$

COROLLARIUM 2

39. At sit $n = 1$, ut aequatio differentialis sit:

$$Pydx + Qydx + Rdy = 0$$

$$\frac{Pdx + Qdx}{R} + \frac{dy}{y} = 0,$$

cuius integralis manifesto est

$$\int \frac{(P+Q)dx}{R} + ly = \text{Const.}$$

SCHOLION

40. Caeterum hoc problema ex antecedente facile ded
enim aequatio differentialis proposita per y^n, et habebitur:

$$Pdx + Qy^{1-n}dx + Ry^{-n}dy = 0\,.$$

Ponatur $y^{1-n} = z$, erit $(1-n)\,y^{-n}dy = dz$, sicque aequatio

$$Pdx + Qzdx + \frac{1}{1-n}Rdz = 0\,,$$

quae cum aequatione problematis praecedentis convenit. Cu
aequationes referendae sint ad casum, quo altera variabili
unam dimensionem ascendit, hunc methodo hac per multipli
mus. Pergo itaque ad alterum genus aequationum differenti
arum, quas etiam hac methodo tractari posse constat. Ad h
quo natura functionum homogenearum continetur, praemitt
quidem operationem ex primis principiis petere volumus.

LEMMA

41. Si V fuerit functio homogenea, in qua binae variabi
n dimensiones constituant, eius differentiale

$$dV = Pdx + Qdy$$

ita erit comparatum, ut sit[1]

$$Px + Qy = nV\,.$$

DEMONSTRATIO

Ponatur $y = xz$, et functio V induet huiusmodi formam
quapiam functione ipsius z tantum. Hinc ergo erit

1) Cf. Commentationem 44 huius voluminis, § 22—23, p. 48.

est, ut sit
$$nx^{n-1}Z = P + Qz,$$
que multiplicando:
$$nx^n Z = nV = Px + Qxz = Px + Qy,$$
$+ Qy = nV.$

COROLLARIUM 1

ergo habemus duas aequationes:
$$dV = Pdx + Qdy \quad \text{et} \quad nV = Px + Qy,$$
unctiones P et Q definiri poterunt; reperietur enim:
$$P = \frac{ydV - nVdy}{ydx - xdy} \quad \text{et} \quad Q = \frac{nVdx - xdV}{ydx - xdy}.$$

COROLLARIUM 2

oties ergo V est functio homogenea n dimensionum, toties ob
$$P = \left(\frac{dV}{dx}\right) \text{ et } Q = \left(\frac{dV}{dy}\right)$$
$$\left(\frac{dV}{dx}\right) = \frac{ydV - nVdy}{ydx - xdy} \text{ et } \left(\frac{dV}{dy}\right) = \frac{nVdx - xdV}{ydx - xdy},$$
m est, in his fractionibus differentialia se mutuo tollere, seu utrum-
torem fore per $ydx - xdy$ divisibilem.

PROBLEMA 6

oposita aequatione differentiali
$$Mdx + Ndy = 0,$$
et N sint functiones homogeneae ipsarum x et y, eiusdem ambae
m numeri, invenire multiplicatorem, qui eam aequationem reddat
m.

Sit n numerus dimensionum, utrique functioni M et N conv
que per paragraphum praecedentem

$$\left(\frac{dM}{dy}\right) = \frac{nMdx - xdM}{ydx - xdy} \quad \text{et} \quad \left(\frac{dN}{dx}\right) = \frac{ydN - nNdy}{ydx - xdy}$$

ideoque

$$\left(\frac{dM}{dy}\right) - \left(\frac{dN}{dx}\right) = \frac{n(Mdx + Ndy) - xdM - ydN}{ydx - xdy}.$$

Jam facile colligere licet dari multiplicatorem, qui etiam sit functio
ipsarum x et y. Sit ergo L talis functio homogenea m dimensioni
in § 19 ponatur

$$dL = Pdx + Qdy,$$

erit [§ 42]

$$P = \frac{ydL - mLdy}{ydx - xdy} \quad \text{et} \quad Q = \frac{mLdx - xdL}{ydx - xdy}$$

hincque, cum esse oporteat per § 19

$$\frac{NP - MQ}{L} = \left(\frac{dM}{dy}\right) - \left(\frac{dN}{dx}\right),$$

obtinebitur utrinque per $ydx - xdy$ multiplicando:

$$\frac{NydL - mLNdy - mLMdx + MxdL}{L} = n(Mdx + Ndy) - xdM$$

unde elicitur:

$$\frac{dL}{L} = \frac{(m+n)(Mdx + Ndy) - xdM - ydN}{Mx + Ny},$$

quae formula manifesto fit integrabilis posito $m + n = -1$, q

$$lL = -l(Mx + Ny).$$

Quam ob rem multiplicator quaesitus habebitur

$$L = \frac{1}{Mx + Ny}.$$

COROLLARIUM 1

45. Proposita igitur aequatione differentiali homogenea Mdx
ea facillime ad integrabilitatem reducetur, propterea quod formu

s, cuius integrale, per methodum supra traditam inventum, dabit integralem quaesitam.

COROLLARIUM 2

casu tantum incommodum oritur, ubi fit $Mx + Ny = 0$, veluti atione $ydx - xdy = 0$, quae dividi deberet per

$$xy - xy = 0 \cdot xy.$$

us divisoris multiplum quodcunque aeque satisfacit, divisor xy ficiet, quemadmodum per se est perspicuum.

SCHOLION

issima est methodus, qua sagacissimus *Ioh. Bernoullius* olim tiones differentiales homogeneas ad separabilitatem variabilium nit. Proposita scilicet huiusmodi aequatione

$$Mdx + Ndy = 0,$$

N sint functiones homogeneae n dimensionum, ponere iubet facto functiones M et N huiusmodi formas induent, ut sit

$$M = x^n U \quad \text{et} \quad N = x^n V,$$

U et V functionibus ipsius u tantum. Aequatio ergo proposita abibit in hanc:

$$Udx + Vdy = 0.$$

it $dy = udx + xdu$, habebimus

$$Udx + Vudx + Vxdu = 0,$$

$U + Vu)$ divisa fit separabilis, seu haec forma

$$\frac{(U + Vu)dx + Vxdu}{x(U + Vu)}$$

At est

$$(U + Vu)dx + Vxdu = \frac{1}{x^n}(Mdx + Ndy)$$

$$\frac{Mdx + Ndy}{x(M + Nu)} = \frac{Mdx + Ndy}{Mx + Ny} \text{ ob } ux = y.$$

Expositis igitur his duobus aequationum generibus, quae per id
catores integrabiles reddi possunt, videamus, ad quaenam alia
methodus extendi possit: ac primo quidem observo, omnes ae
rentiales, quae aliis methodis integrari possunt, etiam hac m
neum multiplicatorem tractari posse, id quod in sequente pr
explicabitur.

PROBLEMA 7

48. Proposita aequatione differentiali $Mdx + Ndy = 0$, s
eius integralis aequatio completa, assignare omnes multiplicat
tionem differentialem reddant integrabilem.

SOLUTIO

Cum aequatio integralis completa involvat quantitate
arbitrariam C, quae in aequatione differentiali non inest,
implicata, quaeratur eius valor per resolutionem aequationis,
eritque V functio ipsarum x et y, quae insuper constantes a
rentialis in se complectetur. Tum ista aequatio $C = V$ differ
prodibit $0 = dV$. Ac iam necesse est, ut dV divisorem habeat i
differentialem propositam. Sit itaque

$$dV = L\,(Mdx + Ndy),$$

eritque L multiplicator idoneus, qui aequationem differentia
reddit integrabilem. Deinde cum, denotante Z functionem qua
V, sit etiam formula

$$ZdV = LZ\,(Mdx + Ndy)$$

integrabilis, expressio LZ omnes multiplicatores includet,
differentialis proposita $Mdx + Ndy = 0$ fit integrabilis.

COROLLARIUM 1

49. Quoties ergo aequationis differentialis $Mdx + Nd$
completum assignari potest, toties non solum unus, sed plane
catores definire licet, quibus ea aequatio integrabilis reddatur

nventa, hinc methodus hactenus tradita, quae ad duo tantum enera adhuc est applicata, non mediocriter amplificari poterit.

SCHOLION

n tamen, nisi ad specialissima exempla descendere velimus, fferentiales, quarum integralia completa assignare licet, ad erum reducuntur. Ac primo quidem occurrunt aequationes rimi gradus in hac forma contentae

$$dx\,(\alpha + \beta x + \gamma y) + dy\,(\delta + \varepsilon x + \zeta y) = 0,$$

e ad homogeneas revocantur, etiam hac methodo per multipli- i poterunt. Deinde memoratu digna est haec forma

$$dy + Py\,dx + Qyy\,dx = R\,dx,$$

t unus valor singularis satisfaciens, ex eo integrale completum quo his casibus multiplicatores idoneos assignare licebit. Tertio i merentur casus huius aequationis

$$dy + yy\,dx = ax^m\,dx,$$

Riccatiana dictae, quibus ea ad separabilitatem reduci potest. unt casus huius aequationis

$$y\,dy + Py\,dx = Q\,dx,$$

integrabiles, ad multiplicatorum investigationem sunt accom- nova patefiet via ex data multiplicatorum forma eas aequationes ae per eos fiant integrabiles, unde fortasse haud spernenda ementa haurire licebit.

PROBLEMA 8

sita aequatione differentiali primi gradus:

$$(\alpha + \beta x + \gamma y)\,dx + (\delta + \varepsilon x + \zeta y)\,dy = 0,$$

plicatores, qui eam reddant integrabilem.

Reducatur haec aequatio ad homogeneitatem ponendo:

$$x = t + f \quad \text{et} \quad y = u + g,$$

ut prodeat

$$(\alpha + \beta f + \gamma g + \beta t + \gamma u)\, dt + (\delta + \varepsilon f + \zeta g + \varepsilon t + \zeta u)\, du$$

quae posito

$$\alpha + \beta f + \gamma g = 0 \quad \text{et} \quad \delta + \varepsilon f + \zeta g = 0,$$

unde quantitates f et g determinantur, utique fit homogenea, scilic

$$(\beta t + \gamma u)\, dt + (\varepsilon t + \zeta u)\, du = 0;$$

ideoque per multiplicatorem

$$\frac{1}{\beta tt + (\gamma + \varepsilon) tu + \zeta uu}$$

integrabilis redditur. Hinc inventis litteris f et g aequatio proposita evadet, si dividatur per

$$\beta (x - f)^2 + (\gamma + \varepsilon)(x - f)(y - g) + \zeta (y - g)^2,$$

seu per

$$\beta xx + (\gamma + \varepsilon) xy + \zeta yy - (2\beta f + \gamma g + \varepsilon g)\, x - (2\zeta g + \gamma f$$
$$+ \beta ff + (\gamma + \varepsilon) fg + \zeta gg.$$

Cum autem sit

$$f = \frac{\alpha\zeta - \gamma\delta}{\gamma\varepsilon - \beta\zeta} \quad \text{et} \quad g = \frac{\beta\delta - \alpha\varepsilon}{\gamma\varepsilon - \beta\zeta},$$

prodibit divisor quaesitus:

$$\beta xx + (\gamma + \varepsilon) xy + \zeta yy + \frac{\alpha\gamma\delta - \alpha\alpha\zeta + \alpha\delta\varepsilon - \beta\delta\delta}{\gamma\varepsilon - \beta\zeta}$$
$$+ \frac{-2\alpha\beta\zeta + \beta\gamma\delta - \beta\delta\varepsilon + \alpha\gamma\varepsilon + \alpha\varepsilon\varepsilon}{\gamma\varepsilon - \beta\zeta}\, x + \frac{-2\beta\delta\zeta + \alpha\varepsilon\zeta - \alpha\gamma\zeta + \gamma\delta}{\gamma\varepsilon - \beta\zeta}$$

Invento autem uno divisore, seu multiplicatore, ex eo reperientur possibiles.

COROLLARIUM 1

53. Forma ergo divisoris, per quem aequatio differentialis

$$(\alpha + \beta x + \gamma y)\, dx + (\delta + \varepsilon x + \zeta y)\, dy = 0$$

tegrabilis, est

$$\beta xx + (\gamma + \varepsilon)\, yx + \zeta yy + Ax + By + C,$$

ntes A, B, C supra sunt definitae.

COROLLARIUM 2

un divisor inventus etiam satisfaciat, si per $\gamma\varepsilon - \beta\zeta$ multiplicetur, n, quo $\beta\zeta = \gamma\varepsilon$, divisorem fore

$$+ \beta\gamma\delta - \alpha\beta\zeta)\, x + (\gamma\gamma\delta - \alpha\gamma\zeta + \alpha\varepsilon\zeta - \beta\delta\zeta)\, y + \alpha\gamma\delta - \alpha\alpha\zeta + \alpha\delta\varepsilon - \beta\delta\delta$$

o $\beta = mf$, $\gamma = nf$, $\varepsilon = mg$, $\zeta = ng$, abit in

$$\delta f)\,(mg - nf)\, x + n\,(\alpha g - \delta f)\,(mg - nf)\, y + (\alpha g - \delta f)\,(\delta m - \alpha n)\,.$$

COROLLARIUM 3

Quare si aequatio proposita fuerit huiusmodi:

$$[\alpha + f(mx + ny)]\, dx + [\delta + g(mx + ny)]\, dy = 0,$$

tur integrabilis, si dividatur per

$$(mg - nf)\,(mx + ny) + \delta m - \alpha n$$

$$mx + ny + \frac{\delta m - \alpha n}{mg - nf}.$$

erit $mg - nf = 0$, aequatio proposita iam ipsa est integrabilis.

PROBLEMA 9

. Proposita hac aequatione differentiali:

$$dy + Pydx + Qyydx + Rdx = 0,$$

Q et R sint functiones ipsius x tantum, si constet, huic aequationi satis- $y = v$, existente v functione ipsius x, invenire multiplicatores, qui istam tionem reddant integrabilem.

$$dv + Pvdx + Qvvdx + Rdx = 0;$$

si ergo ponatur $y = v + \frac{1}{z}$, habebitur

$$-\frac{dz}{zz} + \frac{Pdx}{z} + \frac{2Qvdx}{z} + \frac{Qdx}{zz} = 0$$

sive

$$dz - (P + 2Qv)\,zdx - Qdx = 0,$$

quae integrabilis redditur per multiplicatorem

$$e^{\int (P+2Qv)dx}.$$

Hic ergo multiplicator per zz multiplicatus conveniet aequ
Cum ergo sit $z = \frac{1}{y-v}$ multiplicator aequationem proposi
reddens, erit:

$$\frac{1}{(y-v)^2}e^{-\int (P+2Qv)dx}.$$

Sit brevitatis gratia

$$e^{\int (P+2Qv)dx} = S.$$

Quia aequationis

$$dz - (P + 2Qv)\,zdx - Qdx = 0$$

integrale est

$$Sz - \int QSdx = \text{Const.},$$

omnes multiplicatores quaesiti continebuntur in hac forma:

$$\frac{S}{(y-v)^2}\ \text{funct.}\left(\frac{S}{y-v} - \int QSdx\right),$$

ubi per hypothesin v est functio cognita ipsius x, ideoque etiar

COROLLARIUM I

57. Multiplicator ergo, qui primum se obtulit, est

$$\frac{S}{(y-v)^2},$$

tum vero etiam multiplicator erit

$$\frac{S}{S(y-v) - (y-v)^2\int QSdx},$$

COROLLARIUM 2

enim S est quantitas exponentialis, fieri potest, ut $\int QSdx$ huius-
m ST induat existente T functione algebraica, quo casu multipli-

$$\frac{1}{y-v-(y-v)^2T}=\frac{1}{(y-v)(1-Ty+Tv)}$$

ebraicus, quod in priori forma fieri nequit.

COROLLARIUM 3

m his duobus casibus multiplicator sit fractio, in cuius solum
rem variabilis y ingreditur, ibique ultra quadratum non ascendat,
les alii huiusmodi multiplicatores exhiberi possunt: Sit enim
, et fractionis $\frac{S}{(y-v)^2}$ denominatorem multiplicare licebit per

$$A+B\left(\frac{S}{y-v}-V\right)+C\left(\frac{S}{y-v}-V\right)^2,$$

generalior multiplicatoris forma:

$$\frac{S}{\ldots v)^2+BS(y-v)-BV(y-v)^2+CSS-2CSV(y-v)+CVV(y-v)^2}$$

$$\frac{S}{\ldots-(2Av-BS-2BVv+2CSV+2CVVv)y+Avv-BSv-BVvv+CSS+2CSVv+CV^2v^2}$$

COROLLARIUM 4

uodsi ergo haec formula

$$\frac{dy+Pydx+Qyydx+Rdx}{Lyy+My+N}$$

grabilis, denominator ita debet esse comparatus, ut sit

$$\ldots A-BV+CVV,\ SM=S(B-2CV)-2v(A-BV+CVV)$$

et $V = \int QSdx$.

PROBLEMA 10

61. Proposita aequatione differentiali praecedente:

$$dy + Pydx + Qyydx + Rdx = 0$$

invenire functiones L, M et N ipsius x, ut ea per formulam

$$Lyy + My + N$$

divisa fiat integrabilis.

SOLUTIO

Cum igitur integrabilis esse debeat haec formula:

$$\frac{dy + dx(Py + Qyy + R)}{Lyy + My + N},$$

per proprietatem generalem esse oportet, postquam per

$$(Lyy + My + N)^2$$

multiplicaverimus:

$$-\frac{yydL}{dx} - \frac{ydM}{dx} - \frac{dN}{dx} = \begin{matrix} + QMyy - 2RLy + N \\ - PLyy + 2QNy - R \end{matrix}$$

Unde pro determinatione functionum L, M et N has consequim

$$\begin{aligned} \text{I.}\ dL &= PLdx - QMdx \\ \text{II.}\ dM &= 2RLdx - 2QNdx \\ \text{III.}\ dN &= RMdx - PNdx, \end{aligned}$$

ex quarum prima deducimus:

$$M = \frac{PL}{Q} - \frac{dL}{Qdx}$$

et ex secunda:

$$N = \frac{RL}{Q} - \frac{dM}{2Qdx},$$

qui valores pro M et N in tertia substituti, dant:

$$dN = \frac{PdM}{2Q} - \frac{RdL}{Q}.$$

n sit, sumto differentiali dx constante,

$$dM = \frac{PdL + LdP}{Q} - \frac{PLdQ}{QQ} - \frac{ddL}{Qdx} + \frac{dQdL}{QQdx},$$

$$\ldots = \frac{RL}{Q} - \frac{PdL}{2QQdx} - \frac{LdP}{2QQdx} + \frac{PLdQ}{2Q^3dx} + \frac{ddL}{2QQdx^2} - \frac{dQdL}{2Q^3dx^2}$$

$$\ldots = \frac{PPdL}{2QQ} + \frac{PLdP}{2QQ} - \frac{PPLdQ}{2Q^3} - \frac{PddL}{2QQdx} + \frac{PdQdL}{2Q^3dx} - \frac{RdL}{Q},$$

illius differentiali debet aequari, unde fit:

$$\begin{aligned}&QQd^3L - 3QdQddL - PPQQdLdx^2 - 2QQdPdLdx\\ &3dQ^2dL + 2PQdQdLdx - QdLddQ + 4Q^3RdLdx^2\\ &PQQLdPdx^2 + PPQLdQdx^2 - QQLdxddP + PQLdxddQ\\ &3QLdPdQdx - 3PLdQ^2dx + 2Q^3LdRdx^2 - 2Q^2RLdQdx^2.\end{aligned}$$

m aequatio si per $\frac{L}{Q^4}$ multiplicetur, integrari poterit, eritque eius

$$\ldots \frac{ddL}{QQ} - \frac{LdLdQ}{Q^3} - \frac{dL^2}{2QQ} - \frac{PPLLdx^2}{2QQ} - \frac{LLdPdx}{QQ} + \frac{PLLdQdx}{Q^3} + \frac{2RLLdx^2}{Q},$$

nc formam abit:

$$\begin{aligned}\ldots{}^2 = 2QLddL - 2LdLdQ - QdL^2 - PPQLLdx^2 - 2QLLdPdx\\ + 2PLLdQdx + 4QQRLLdx^2.\end{aligned}$$

natur $L = zz$, aequatio induet hanc formam:

$$\ldots 4Qddz - 4dQdz - z(PPQdx^2 + 2QdPdx - 2PdQdx - 4QQRdx^2).$$

COROLLARIUM 1

notics ergo per problema praecedens valor ipsius L assignari potest, atio differentialis tertii ordinis hic inventa, et ea secundi ordinis, ad n reduxi, generaliter resolvi poterit: quae resolutio, cum alias foret , probe est notanda.

COROLLARIUM 2

cilicet si v fuerit eiusmodi functio ipsius x, quae loco y posita, satis- uationi

statuaturque $V = \int QSdx$, quo facto erit pro nostra aeq
tertii ordinis

$$L = \frac{A - BV + CVV}{S},$$

qui valor cum tres constantes arbitrarias complectatur, a
tionis integrale completum.

COROLLARIUM 3

63. Si sit $P = 0$, $Q = 1$ et R functio quaecumque
differentialis tertii gradus hanc accipiet formam:

$$0 = d^3 L + 4RdLdx^2 + 2LdRdx^2,$$

pro cuius ergo integrali completo inveniendo, quaeratur pr
quae sit $= v$, quae satisfaciat huic aequationi

$$dv + vvdx + Rdx = 0;$$

tum ponatur

$$V = \int e^{-2\int vdx} dx,$$

eritque

$$L = (A - BV + CVV)\, e^{+2\int vdx}.$$

COROLLARIUM 4

64. Idem ergo integrale satisfaciet huic aequationi
gradus:

$$2Edx^2 = 2LddL - dL^2 + 4RLLdx^2$$

et, posito $L = zz$, etiam huic:

$$\frac{Edx^2}{2z^3} = ddz + Rzdx^2,$$

pro qua itaque est

$$z = e^{+\int vdx} \sqrt{(A - BV + CVV)}.$$

Omnino animadvorti meretur haec integratio, quippe quae ex aliis s vix quidem praestari potest. Hinc autem adipiscimur[1]) integrationem am sequentis aequationis differentio-differentialis satis late patentis:

$$ddz + Sdxdz + Tzdx^2 = \frac{Edx^2}{z^3}e^{-2\int Sdx}.$$

nempe quaeratur valor ipsius v ex hac aequatione differentiali primi

$$dv + vvdx + Svdx + Tdx = 0,$$

vento ponatur brevitatis ergo

$$V = \int e^{-2\int vdx - \int Sdx}dx$$

$$z = e^{\int vdx}\sqrt{(A + BV + CVV)},$$

e constantes arbitrariae A, B, C ita accipiantur, ut sit

$$AC - \frac{1}{4}BB = E,$$

adhuc duae constantes arbitrio nostro relinquuntur, uti natura intenis completae postulat.

EXEMPLUM 1

36. *Proposita sit haec aequatio differentialis*

$$dy + ydx + yydx - \frac{dx}{x} = 0,$$

multiplicatores, qui eam reddant integrabilem, investigari oporteat.

Erit ergo, Problema 9 huc transferendo,

$$P = 1,\ Q = 1 \text{ et } R = -\frac{1}{x},$$

uia aequationi satisfacit valor $y = \frac{1}{x}$, erit $v = \frac{1}{x}$. Quare fiet

$$S = e^{-\int\left(1+\frac{2}{x}\right)dx} = \frac{1}{xx}e^{-x}$$

1) Si in formulis § 63 et 64 ponuntur

$$ze^{\int\frac{S}{2}dx} \text{ loco } z,\ v + \frac{S}{2} \text{ loco } v \text{ et } T = \frac{dS}{2dx} + \frac{S^2}{4} + R.$$ H. D.

Hunc autem porro multiplicare licet per functionem qu
formae

$$e^{-x}\frac{1}{x(xy-1)}-\int e^{-x}\frac{dx}{xx};$$

cum vero haec forma integrari nequeat, alii multiplicatores
nequeunt. Ob primum ergo integrabilis est haec forma:

$$e^{-x}\frac{1}{(xy-1)^2}\left(dy+y\,dx+yy\,dx-\frac{dx}{x}\right),$$

cuius, si x capitur constans, integrale est

$$\frac{-e^{-x}}{x(xy-1)}+X,$$

quae differentiata, posito y constante, praebet

$$\frac{e^{-x}dx(xxy+2xy-x-1)}{xx(xy-1)^2}+dX,$$

quod aequari debet alteri membro

$$\frac{e^{-x}}{(xy-1)^2}\left(y\,dx+yy\,dx-\frac{dx}{x}\right),$$

unde fit

$$dX=\frac{e^{-x}dx}{xx(xy-1)^2}(xxyy-2xy+1)=e^{-x}\frac{d}{x}$$

sicque integrale completum nostrae aequationis est

$$\frac{-e^{-x}}{x(xy-1)}+\int e^{-x}\frac{dx}{xx}=\text{Const.}$$

EXEMPLUM 2

67. Invenire multiplicatores idoneos, qui reddant h
integrabilem[1]):

1) Vide notam 2 p. 309.

us singularis huic aequationi satisfaciens est

$$y = \frac{k + \gamma x}{\alpha + \beta x + \gamma xx} = v$$

stente

$$k = \tfrac{1}{2}\beta \pm \sqrt{(\tfrac{1}{4}\beta\beta - \alpha\gamma + a)}.$$

n nunc sit $P = 0$ et $Q = 1$, erit

$$S = e^{\int \frac{2k\,dx + 2\gamma x\,dx}{\alpha + \beta x + \gamma xx}}$$

posito brevitatis gratia

$$\pm \sqrt{(\tfrac{1}{4}\beta\beta - \alpha\gamma + a)} = \tfrac{1}{2} n$$

t

$$S = \frac{1}{\alpha + \beta x + \gamma xx} e^{\int \frac{n\,dx}{\alpha + \beta x + \gamma xx}}$$

$$\int S\,dx = -\frac{1}{n} e^{\int \frac{n\,dx}{\alpha + \beta x + \gamma xx}}.$$

ltiplicator ergo primum inventus est

$$e^{\int \frac{n\,dx}{\alpha + \beta x + \gamma xx}} \cdot \frac{\alpha + \beta x + \gamma xx}{((\alpha + \beta x + \gamma xx)\,y - k - \gamma x)^2},$$

i porro duci potest in functionem quamcunque huius quantitatis

$$e^{\int \frac{n\,dx}{\alpha + \beta x + \gamma xx}} \left(\frac{1}{(\alpha + \beta x + \gamma xx)\,y - k - \gamma x} + \frac{1}{n}\right).$$

ucatur ergo in

$$e^{\int \frac{n\,dx}{\alpha + \beta x + \gamma xx}} \cdot \frac{(\alpha + \beta x + \gamma xx)\,y - k - \gamma x}{(\alpha + \beta x + \gamma xx)\,y + n - k - \gamma x}$$

prodibit multiplicator algebraicus:

$$\frac{\alpha + \beta x + \gamma xx}{((\alpha + \beta x + \gamma xx)\,y - k - \gamma x)\,((\alpha + \beta x + \gamma xx)y + n - k - \gamma x)},$$

ui reducitur ad hanc formam:

$$= e^{-x}\frac{}{(xy-1)^2}.$$

Hunc autem porro multiplicare licet per functionem quam formae

$$e^{-x}\frac{1}{x(xy-1)} - \int e^{-x}\frac{dx}{xx};$$

cum vero haec forma integrari nequeat, alii multiplicatores i nequeunt. Ob primum ergo integrabilis est haec forma:

$$e^{-x}\frac{1}{(xy-1)^2}\left(dy + ydx + yy\,dx - \frac{dx}{x}\right),$$

cuius, si x capitur constans, integrale est

$$\frac{-e^{-x}}{x(xy-1)} + X,$$

quae differentiata, posito y constante, praebet

$$\frac{e^{-x}dx(xxy+2xy-x-1)}{xx(xy-1)^2} + dX,$$

quod aequari debet alteri membro

$$\frac{e^{-x}}{(xy-1)^2}\left(ydx + yydx - \frac{dx}{x}\right),$$

unde fit

$$dX = \frac{e^{-x}dx}{xx(xy-1)^2}(xxyy - 2xy + 1) = e^{-x}\frac{dx}{xx};$$

sicque integrale completum nostrae aequationis est

$$\frac{-e^{-x}}{x(xy-1)} + \int e^{-x}\frac{dx}{xx} = \text{Const.}$$

EXEMPLUM 2

67. Invenire multiplicatores idoneos, qui reddant han integrabilem[1]):

1) Vide notam 2 p. 300.

$$S = e^{\int \frac{2kdx + 2\gamma xdx}{\alpha + \beta x + \gamma xx}}$$

ito brevitatis gratia

$$1 + \sqrt{\tfrac14\beta\beta - \alpha\gamma + \alpha} = \tfrac12 n$$

$$S = \frac{1}{\alpha + \beta x + \gamma xx} e^{\int \frac{ndx}{\alpha + \beta x + \gamma xx}}$$

$$\int S dx = \frac{1}{n} e^{\int \frac{ndx}{\alpha + \beta x + \gamma xx}},$$

plicator ergo primum inventus est

$$e^{\int \frac{ndx}{\alpha + \beta x + \gamma xx}} \cdot \frac{\alpha + \beta x + \gamma xx}{((\alpha + \beta x + \gamma xx)y - k - \gamma x)^2},$$

orro duci potest in functionem quamcunque huius quantitatis

$$e^{\int \frac{ndx}{\alpha + \beta x + \gamma xx}} \left(\frac{1}{(\alpha + \beta x + \gamma xx)y - k - \gamma x} + \frac{1}{n} \right).$$

atur ergo in

$$e^{\int \frac{ndx}{\alpha + \beta x + \gamma xx}} \cdot \frac{(\alpha + \beta x + \gamma xx)y - k - \gamma x}{(\alpha + \beta x + \gamma xx)y + n - k - \gamma x}$$

prodibit multiplicator algebraicus:

$$\frac{\alpha + \beta x + \gamma xx}{((\alpha + \beta x + \gamma xx)y - k - \gamma x)((\alpha + \beta x + \gamma xx)y + n - k - \gamma x)},$$

reducitur ad hanc formam:

$$\overline{(\alpha+\beta x+\gamma xx)\left(y-\frac{2\gamma x+\beta+\sqrt{(\beta\beta-4\alpha\gamma+4a)}}{2(\alpha+\beta x+\gamma xx)}\right)\left(y-\frac{2\gamma x+\beta-\sqrt{(}}{2(\alpha+}\right.}$$

Aequationis autem integrale completum est

$$e^{-\int\frac{ndx}{\alpha+\beta x+\gamma xx}}\frac{(\alpha+\beta x+\gamma xx)y+n-k-\gamma x}{(\alpha+\beta x+\gamma xx)y-k-\gamma x}=\mathrm{C}$$

existente $n=\sqrt{(\beta\beta-4\alpha\gamma+4a)}$ et $k=\frac{\beta+n}{2}$.

Ex quo aequatio integralis completa erit

$$e^{-\int\frac{ndx}{\alpha+\beta x+\gamma xx}}\cdot\frac{2(\alpha+\beta x+\gamma xx)y+n-\beta-2\gamma x}{2(\alpha+\beta x+\gamma xx)y-n-\beta-2\gamma x}=$$

cuius indoles est manifesta, dummodo

$$n=\sqrt{(\beta\beta-4\alpha\gamma+4a)}$$

sit numerus realis.

Quodsi autem valor ipsius n sit imaginarius, puta $n=$

$$e^{p\sqrt{-1}}=\cos.p+\sqrt{-1}\sin.p,$$

aequatio integralis ita ad realitatem perduci potest. Sit

$$-m\int\frac{dx}{\alpha+\beta x+\gamma xx}=p\quad\text{et}\quad 2(\alpha+\beta x+\gamma xx)y-\beta$$

eritque ea:

$$(\cos.p+\sqrt{-1}\sin.p)\cdot\frac{q+m\sqrt{-1}}{q-m\sqrt{-1}}=\text{Const.}=A+$$

hinc fit:

$$q\cos.p-m\sin.p+(m\cos.p+q\sin.p)\sqrt{-1}=Aq+Bm+$$

aequentur seorsim membra realia et imaginaria:

$$q\cos.p-m\sin.p=Aq+Bm,\quad m\cos.p+q\sin.p$$

quae duae aequationes congruunt, si capiatur $AA+B$ constans arbitraria $A=\cos.\theta$, ut sit $B=\sin.\theta$ et casu, quo $=m\sqrt{-1}$, aequatio realis erit

$$q\cos.p-m\sin.p=q\cos.\theta+m\sin.\theta\quad\text{seu}\quad q=\frac{m(\sin.p+\sin}{\cos.p-\cos}$$

$$dy + yydx + \frac{(\ldots)}{4(\alpha + \beta x + \gamma xx)^2} = 0,$$

$$p = \int \frac{-mdx}{\alpha + \beta x + \gamma xx},$$

...lis completa est

$$2(\alpha + \beta x + \gamma xx)y = \beta + 2\gamma x + m \cot \frac{\theta - p}{2}$$

$$y = \frac{\frac{1}{2}\beta + \gamma x + \frac{1}{2} m \cot. \frac{\theta - p}{2}}{\alpha + \beta x + \gamma xx},$$

... $-\zeta$, et habebitur

$$y = \frac{\frac{1}{2}\beta + \gamma x + \frac{1}{2} m \operatorname{tang.} \frac{\zeta + p}{2}}{\alpha + \beta x + \gamma xx}.$$

...a notandum est, integrale speciale, ex quo haec omnia deduxi-...inarium, quo tamen non obstante inde integrale completum in...ibere licuit.

EXEMPLUM 3

...ita aequatione Riccatiana

$$dy + yydx - ax^m dx = 0,$$

...ponentis m, quibus eam separare licet, invenire multiplicatores

...valor aequationi satisfaciens, et cum sit

$$P = 0,\ Q = 1 \text{ et } R = -ax^m,$$

...ltiplicator, aequationem integrabilem reddens,

$$e^{-2\int vdx} \frac{1}{(y-v)^2},$$

...equatio multiplicetur, integrale completum fit

$$e^{-2\int vdx} \frac{1}{y-v} - \int e^{-2\int vdx} dx = \text{Const.}$$

$$e^{-2\int v dx} \frac{}{(y-v)^2}.$$

Hinc si ponatur

$$\int e^{-2\int v dx} dx = V,$$

omnes multiplicatores in hac forma

$$\frac{1}{Lyy + My + N}$$

contenti obtinebuntur [§ 60], si capiatur:

$$\begin{aligned} L &= e^{2\int v dx}(A - BV + CVV) \\ M &= B - 2CV - 2ve^{2\int v dx}(A - BV + CVV) \\ N &= Ce^{-2\int v dx} - v(B - 2CV) + vve^{2\int v dx}(A - BV \end{aligned}$$

Verum hic valor ipsius L simul est integrale completum h differentialis tertii gradus:

$$0 = d^3L - 4ax^m dLdx^2 - 2maLx^{m-1}dx^3$$

hincque etiam huius secundi gradus:

$$Edx^2 = 2LddL - dL^2 - 4aLLx^m dx^2$$

existente

$$E = 4AC - BB.$$

SCHOLION

69. Re attentius perpensa aequationem differentialem te methodo directa resolvi, eiusque integrale completum ide assignatum, elici posse deprehendi. Sit enim proposita haec a

$$d^3L + 4RdLdx^2 + 2LdRdx^2 = 0,$$

ubi R sit functio quaecunque ipsius x, sumto differentiali d quaero functionem ipsius x, per quam ista aequatio mu integrabilis. Sit S ista functio, et aequationis

$$Sd^3L + 4SRdLdx^2 + 2SLdRdx^2 = 0$$

integrale erit

$$SddL - dSdL + L(ddS + 4SRdx^2) = 2Cd$$

do sit
$$d^3S + 2SdRdx^2 + 4RdSdx^2 = 0.$$
scilicet quemvis valorem particulariter satisfacientem sumsisse. At quatio, per S multiplicata, neglecta constante, dat integrale:
$$SddS - \tfrac{1}{2}dS^2 + 2SSRdx^2 = 0.$$
r $S = e^{2\int vdx}$, eritque
$$2dv + 2vvdx + 2Rdx = 0,$$
negotium huc redit, ut pro v saltem valor particularis investigetur, qui ciat huic aequationi differentiali primi gradus:
$$dv + vvdx + Rdx = 0,$$
igitur tanquam concessum assumo. Hinc nostra aequatio semel integrata b $S = e^{2\int vdx}$,
$$ddL - 2vdxdL + L(2dvdx + 4vvdx^2 + 4Rdx^2) = 2Ce^{-2\int vdx}dx^2.$$
gitur, ob
$$Rdx = -dv - vvdx,$$
mus
$$ddL - 2vdxdL - 2Ldxdv = 2Ce^{-2\int vdx}dx^2,$$
ntegrale manifesto est:
$$dL - 2Lvdx = Bdx + 2Cdx\int e^{-2\int vdx}dx$$
r $e^{-2\int vdx}$ denuo multiplicando integrale, prodibit
$$e^{-2\int vdx}L = A + B\int e^{-2\int vdx}dx + 2C\int e^{-2\int vdx}dx\int e^{-2\int vdx}dx.$$
re si brevitatis gratia ponatur $\int e^{-2\int vdx}dx = V$, habebimus
$$L = e^{2\int vdx}(A + BV + 2CVV)$$
sus uti ante invenimus.

PROBLEMA 11

70. Proposita aequatione Riccatiana
$$dy + yydx = ax^m dx,$$
enire eius integralia particularia, casibus, quibus ea separabilis existit[1]).

H. D

1) Vide notam 1 p. 17.

$$dy + yydx - ccx^{-4n}dx = 0.$$

Cum enim quaestio circa integralia particularia versetur, nihil interest, ea sint realia, nec ne. Quo autem facilius, et una quasi operatione, hos quibus y per functionem ipsius x exprimere licet, eliciamus: statuamus

$$y = cx^{-2n} + \frac{dz}{zdx}$$

et sumto dx constante, nanciscemur hanc aequationem differentialem gradus:

$$-2ncx^{-2n-1}dx + \frac{ddz}{zdx} + \frac{2cx^{-2n}dz}{z} = 0,$$

seu

$$\frac{ddz}{dx^2} + \frac{2cdz}{x^{2n}dx} - \frac{2ncz}{x^{2n+1}} = 0,$$

cuius valor fingatur:

$$z = Ax^n + Bx^{3n-1} + Cx^{5n-2} + Dx^{7n-3} + Ex^{9n-4} + \text{etc.},$$

quo debite substituto obtinebimus:

$$\begin{array}{rlrlrl}
0 = & n(n-1)Ax^{n-2} & + & (3n-1)(3n-2)Bx^{3n-3} & + & (5n-2)(5n- \\
+ & 2ncAx^{-n-1} + 2(3n-1)cB & + & 2(5n-2)cC & + & 2(7n- \\
- & 2ncA \qquad - \qquad 2ncB & - & 2ncC & - &
\end{array}$$

unde coefficientes ficti ita determinantur:

$$2(2n-1)cB + n(n-1)A = 0, \qquad B = \frac{-n(n-1)A}{2(2n-1)c}$$

$$2(4n-2)cC + (3n-1)(3n-2)B = 0, \quad C = \frac{-(3n-1)(3n-2}{4(2n-1)c}$$

$$2(6n-3)cD + (5n-2)(5n-3)C = 0, \quad D = \frac{-(5n-2)(5n-3}{6(2n-1)c}$$

etc.

Statim igitur atque unus coefficiens evanescit, sequentes simul omnes eunt, id quod evenit his casibus:

$$n = 0, \quad n = \frac{1}{3}, \quad n = \frac{2}{5}, \quad n = \frac{3}{7}, \quad \text{etc.}$$

$$n = 1, \quad n = \frac{2}{3}, \quad n = \frac{3}{5}, \quad n = \frac{4}{7}, \quad \text{etc.}$$

$$n = \frac{i}{2i \pm 1},$$

aequationis exhiberi potest. Erit enim

$$y = cx^{-2n} + \frac{dz}{z\,dx},$$

$$= Ax^n + Bx^{3n-1} + Cx^{5n-2} + Dx^{7n-3} + Ex^{9n-4} + \text{etc.}$$

o hic valor particularis ipsius y:

$$cx^{-2n} + \frac{nAx^{n-1} + (3n-1)Bx^{3n-2} + (5n-2)Cx^{5n-3} + \text{etc.}}{Ax^n + Bx^{3n-1} + Cx^{5n-2} + \text{etc.}}$$

COROLLARIUM 1

dsi ergo iste valor particularis ipsius y vocetur $= v$, erit aequationis nultiplicator idoneus

$$= e^{-2\int v\,dx} \cdot \frac{1}{(y-v)^2}.$$

ur

$$\int e^{-2\int v\,dx} dx = V,$$

0 et $C = 0$, erit alius factor simplicior [§ 68]

$$\frac{1}{e^{2\int v\,dx}Vyy - (1 + 2ve^{2\int v\,dx}V)y + v + vve^{2\int v\,dx}V}.$$

COROLLARIUM 2

est

$$\int v\,dx = \frac{-c}{(2n-1)x^{2n-1}} + l(Ax^n + Bx^{3n-1} + Cx^{5n-2} + \text{etc.}),$$

$$e^{-2\int v\,dx} = e^{\frac{2c}{(2n-1)x^{2n-1}}} \frac{1}{(Ax^n + Bx^{3n-1} + Cx^{5n-2} + \text{etc.})^2},$$

orro inveniri potest valor ipsius

existente T functione algebraica, erit superior multiplicator

COROLLARIUM 3

73. Invento valore v, seu integrali particulari aequationi
statim habebitur integrale completum eiusdem, quippe quod

$$\frac{e^{-2\int v dx}}{y-v} - \int e^{-2\int v dx} dx = \text{Const.}$$

CASUS 1 quo $n = 0$

74. Pro hac ergo aequatione

$$dy + yydx = ccdx,$$

ob $B = 0$, $C = 0$ etc., erit valor particularis $y = c$. Quare p

$$e^{-2\int v dx} = e^{-2cx} \quad \text{et} \quad V = \int e^{-2\int v dx} dx = -\frac{1}{2c} e^{-2c}$$

unde integrale completum est

$$\frac{e^{-2cx}}{y-c} + \frac{y}{2c} e^{-2cx} = \text{Const.}$$

seu

$$\frac{e^{-2cx}(y+c)}{y-c} = \text{Const.}$$

Porro, ob

$$e^{2\int v dx} V = -\frac{1}{2c} \quad \text{et} \quad v = c,$$

erit multiplicator algebraicus:

$$\frac{1}{-\frac{1}{2c} yy + \frac{1}{2} c},$$

qui reducitur ad

$$\frac{1}{yy - cc}$$

uti per se est perspicuum.

$$dy + yydx = \frac{ccdx}{x^4}$$

$= 0$ etc. erit valor particularis

$$y = \frac{c}{xx} + \frac{1}{x}.$$

,

$$v = \frac{c}{xx} + \frac{1}{x},$$

$$e^{-2\int v dx} = \frac{e^{\frac{2c}{x}}}{xx} \quad \text{et} \quad V = -\frac{1}{2c}e^{\frac{2c}{x}}.$$

completum est

$$\frac{e^{\frac{2c}{x}}}{xxy - x - c} + \frac{e^{\frac{2c}{x}}}{2c} = \text{Const.}$$

$$e^{\frac{2c}{x}} \cdot \frac{xxy - x + c}{xxy - x - c} = \text{Const.}$$

$$e^{2\int v dx} V = -\frac{xx}{2c} \quad \text{et} \quad v = \frac{x + c}{xx},$$

ltiplicator algebraicus:

$$\frac{1}{xxyy - 2xy + 1 - \frac{cc}{xx}} = \frac{1}{(xy - 1)^2 - \frac{cc}{xx}}$$

proposita

$$dy + yydx - \frac{ccdx}{x^4} = 0$$

s, si dividatur per

$$(xy - 1)^2 - \frac{cc}{xx}.$$

$$dy + yy\,dx - ccx^{-\frac{4}{3}}\,dx = 0$$

est $B = -\frac{A}{3c}$, $C = 0$, etc., unde integrale particulare

$$y = cx^{-\frac{2}{3}} + \frac{cx^{-\frac{2}{3}}}{3cx^{\frac{1}{3}} - 1} = \frac{3ccx^{-\frac{1}{3}}}{3cx^{\frac{1}{3}} - 1} = v$$

et

$$e^{-2\int v\,dx} = e^{-6cx^{\frac{1}{3}}} \frac{\text{Const.}}{\left(x^{\frac{1}{3}} - \frac{1}{3c}\right)^2} = e^{-6cx^{\frac{1}{3}}} \frac{1}{\left(3cx^{\frac{1}{3}} - 1\right)^2}$$

hincque

$$V = \int e^{-6cx^{\frac{1}{3}}} \frac{dx}{\left(3cx^{\frac{1}{3}} - 1\right)^2} = -\,e^{-6cx^{\frac{1}{3}}} \frac{3cx^{\frac{1}{3}} + 1}{18c^3\left(3cx^{\frac{1}{3}} - 1\right)}$$

Quare integrale completum est

$$\frac{e^{-6cx^{\frac{1}{3}}}}{\left(3cx^{\frac{1}{3}} - 1\right)^2 y - 3ccx^{-\frac{1}{3}}\left(3cx^{\frac{1}{3}} - 1\right)} + \frac{e^{-6cx^{\frac{1}{3}}}\left(3cx^{\frac{1}{3}} + 1\right)}{18c^3\left(3cx^{\frac{1}{3}} - 1\right)} =$$

sive

$$e^{-6cx^{\frac{1}{3}}} \frac{y\left(1 + 3cx^{\frac{1}{3}}\right) + 3ccx^{-\frac{1}{3}}}{y\left(1 - 3cx^{\frac{1}{3}}\right) - 3ccx^{-\frac{1}{3}}} = \text{Const.}$$

Tum, ob

$$e^{2\int v\,dx} V = \frac{1 - 9ccx^{\frac{2}{3}}}{18c^3},$$

prodibit divisor aequationem integrabilem reddens:

$$\left(y + 3ccx^{-\frac{1}{3}}\right)^2 - 9ccx^{\frac{2}{3}}yy$$

7. Pro hac ergo aequatione

$$dy + yydx - ccx^{-\frac{8}{3}}dx = 0$$

$B = + \frac{A}{3c}$, $C = 0$ etc., unde integrale particulare:

$$y = cx^{-\frac{4}{3}} \frac{2cx^{-\frac{1}{3}}+1}{3cx^{\frac{2}{3}}+x} = \frac{3ccx^{-\frac{2}{3}}+3cx^{-\frac{1}{3}}+1}{3cx^{\frac{2}{3}}+x} = v$$

$$e^{2\int vdx} = e^{6cx^{-\frac{1}{3}}} \cdot \frac{1}{(3cx^{\frac{2}{3}}+x)^2};$$

quo porro elicitur:

$$V = \int \frac{e^{6cx^{-\frac{1}{3}}}dx}{(3cx^{\frac{2}{3}}+x)^2} = \frac{e^{6cx^{-\frac{1}{3}}}(3cx^{\frac{2}{3}}-x)}{18c^3(3cx^{\frac{2}{3}}+x)}.$$

...uro integrale completum erit:

$$e^{6cx^{-\frac{1}{3}}} \frac{(x-3cx^{\frac{2}{3}})y - 1 + 3cx^{-\frac{1}{3}} - 3ccx^{-\frac{2}{3}}}{(x+3cx^{\frac{2}{3}})y - 1 - 3cx^{-\frac{1}{3}} - 3ccx^{-\frac{2}{3}}} = \text{Const.}$$

...um ob

$$e^{2\int vdx}V = \frac{xx - 9ccx^{\frac{4}{3}}}{18c^3}$$

prodit divisor algebraicus aequationem propositam integrabilem reddens:

$$((x+3cx^{\frac{2}{3}})y - 1 - 3cx^{-\frac{1}{3}} - 3ccx^{-\frac{2}{3}})((x-3cx^{\frac{2}{3}})y - 1 + 3cx^{-\frac{1}{3}} - 3ccx^{-\frac{2}{3}})$$

CASUS 5 quo $n = \frac{2}{5}$.

78. Pro hac ergo aequatione

$$dy + yydx - ccx^{-\frac{8}{5}}dx = 0$$

erit

$$dy + yy\,dx - ccx^{-\frac{4}{3}}\,dx = 0$$

est $B = -\frac{A}{3c}$, $C = 0$, etc., unde integrale particulare

$$y = cx^{-\frac{2}{3}} + \frac{cx^{-\frac{2}{3}}}{3cx^{\frac{1}{3}} - 1} = \frac{3ccx^{-\frac{1}{3}}}{3cx^{\frac{1}{3}} - 1} = v$$

et

$$e^{-2\int v\,dx} = e^{-6cx^{\frac{1}{3}}} \frac{\text{Const.}}{\left(x^{\frac{1}{3}} - \frac{1}{3c}\right)^2} = e^{-6cx^{\frac{1}{3}}} \frac{1}{\left(3cx^{\frac{1}{3}} - 1\right)^2}$$

hincque

$$V = \int e^{-6cx^{\frac{1}{3}}} \frac{dx}{\left(3cx^{\frac{1}{3}} - 1\right)^2} = -e^{-6cx^{\frac{1}{3}}} \frac{3cx^{\frac{1}{3}} + 1}{18c^3\left(3cx^{\frac{1}{3}} - \right.}$$

Quare integrale completum est

$$\frac{e^{-6cx^{\frac{1}{3}}}}{\left(3cx^{\frac{1}{3}} - 1\right)^2 y - 3ccx^{-\frac{1}{3}}\left(3cx^{\frac{1}{3}} - 1\right)} + \frac{e^{-6cx^{\frac{1}{3}}}\left(3cx^{\frac{1}{3}} + 1\right)}{18c^3\left(3cx^{\frac{1}{3}} - 1\right)} =$$

sive

$$e^{-6cx^{\frac{1}{3}}} \frac{y\left(1 + 3cx^{\frac{1}{3}}\right) + 3ccx^{-\frac{1}{3}}}{y\left(1 - 3cx^{\frac{1}{3}}\right) - 3ccx^{-\frac{1}{3}}} = \text{Const.}$$

Tum, ob

$$e^{2\int v\,dx}\,V = \frac{1 - 9ccx^{\frac{2}{3}}}{18c^3},$$

prodibit divisor aequationem integrabilem reddens:

$$\left(y + 3ccx^{-\frac{1}{3}}\right)^2 - 9ccx^{\frac{2}{3}}yy$$

o hac ergo aequatione

$$dy + yy\,dx - ccx^{-\frac{8}{3}}dx = 0$$

$-\frac{A}{3c}$, $C = 0$ etc., unde integrale particulare:

$$y = cx^{-\frac{4}{3}}\frac{2cx^{-\frac{1}{3}}+1}{3cx^{\frac{2}{3}}+x} = \frac{3ccx^{-\frac{2}{3}}+3cx^{-\frac{1}{3}}+1}{3cx^{\frac{2}{3}}+x} = v$$

$$e^{-2\int v\,dx} = e^{6cx^{-\frac{1}{3}}}\cdot\frac{1}{(3cx^{\frac{2}{3}}+x)^2};$$

rro elicitur:

$$V = \int\frac{e^{6cx^{-\frac{1}{3}}}dx}{(3cx^{\frac{2}{3}}+x)^2} = \frac{-e^{6cx^{-\frac{1}{3}}}(3cx^{\frac{2}{3}}-x)}{18c^3(3cx^{\frac{2}{3}}+x)}.$$

egrale completum erit:

$$e^{6cx^{-\frac{1}{3}}}\frac{(x-3cx^{\frac{2}{3}})y-1+3cx^{-\frac{1}{3}}-3ccx^{-\frac{2}{3}}}{(x+3cx^{\frac{2}{3}})y-1-3cx^{-\frac{1}{3}}-3ccx^{-\frac{2}{3}}} = \text{Const.}$$

$$e^{2\int v\,dx}V = \frac{xx-9ccx^{\frac{4}{3}}}{18c^3}$$

visor algebraicus aequationem propositam integrabilem reddens:

$$x^{\frac{2}{3}})y-1-3cx^{-\frac{1}{3}}-3ccx^{-\frac{2}{3}})\,((x-3cx^{\frac{2}{3}})y-1+3cx^{-\frac{1}{3}}-3ccx^{-\frac{2}{3}}).$$

CASUS 5 quo $n = \frac{2}{5}$.

Pro hac ergo aequatione

$$dy + yy\,dx - ccx^{-\frac{8}{5}}dx = 0$$

$$y = cx^{-\frac{4}{5}} + \frac{\frac{2}{5}x^{-\frac{8}{5}} - \frac{1}{5}\cdot\frac{3}{5c}x^{-\frac{4}{5}}}{x^{\frac{2}{5}} - \frac{3}{5c}x^{\frac{1}{5}} + \frac{3}{25cc}} = cx^{-\frac{4}{5}} + \frac{10ccx^{-\frac{3}{5}} - 3c}{25ccx^{\frac{2}{5}} - 15cx}$$

seu

$$y = \frac{25c^3x^{-\frac{2}{5}} - 5ccx^{-\frac{3}{5}}}{25ccx^{\frac{2}{5}} - 15cx^{\frac{1}{5}} + 3} = v.$$

Unde integrale completum oritur:

$$e^{-10cx^{\frac{1}{5}}}\cdot\frac{(3 + 15cx^{\frac{1}{5}} + 25ccx^{\frac{2}{5}})y + 5ccx^{-\frac{3}{5}} + 25c^3x^{-\frac{2}{5}}}{(3 - 15cx^{\frac{1}{5}} + 25ccx^{\frac{2}{5}})y + 5ccx^{-\frac{3}{5}} - 25c^3x^{-\frac{2}{5}}} = \mathrm{C}$$

Et si huius fractionis ponatur

numerator $(3 + 15cx^{\frac{1}{5}} + 25ccx^{\frac{2}{5}})y + 5ccx^{-\frac{3}{5}} + 25c^3x^{-\frac{2}{5}} = P$

denominator $(3 - 15cx^{\frac{1}{5}} + 25ccx^{\frac{2}{5}})y + 5ccx^{-\frac{3}{5}} - 25c^3x^{-\frac{2}{5}} =$

erit divisor aequationem propositam integrabilem reddens $= P$

CASUS 6 quo $n = \frac{3}{5}$.

79. Pro hac ergo aequatione

$$dy + yydx - ccx^{-\frac{12}{5}}dx = 0,$$

erit

$$B = \frac{3A}{5c} \text{ et } C = \frac{B}{5c} = \frac{3A}{25cc},\ D = 0 \text{ etc.},$$

hincque integrale particulare prodit:

$$y = cx^{-\frac{6}{5}} + \frac{15ccx^{-\frac{2}{5}} + 12cx^{-\frac{1}{5}} + 3}{25ccx^{\frac{3}{5}} + 15cx^{\frac{4}{5}} + 3x}$$

seu

$$y = \frac{25c^3x^{-\frac{3}{5}} + 30ccx^{-\frac{2}{5}} + 15cx^{-\frac{1}{5}} + 3}{25ccx^{\frac{3}{5}} + 15cx^{\frac{4}{5}} + 3x} = v,$$

completum obtinetur:

$$\frac{15cx^{\frac{4}{5}} + 25ccx^{\frac{3}{5}})\, y - 3 + 15cx^{-\frac{1}{5}} - 30ccx^{-\frac{2}{5}} + 25c^3x^{-\frac{3}{5}}}{15cx^{\frac{4}{5}} + 25ccx^{\frac{3}{5}})\, y - 3 - 15cx^{-\frac{1}{5}} - 30ccx^{-\frac{2}{5}} - 25c^3x^{-\frac{3}{5}}} = \text{Const.}$$

ctore exponentiali $e^{10cx^{-\frac{1}{5}}}$, productum ex numeratore et deno-
bebit divisorem, per quem aequatio proposita divisa evadit

PROBLEMA 12

ante i numerum quemcunque integrum, exhibere resolutionem
onis:

$$dy + yy\,dx - ccx^{\frac{-4i}{2i+1}}dx = 0.$$

SOLUTIO

r sit $n = \frac{i}{2i+1}$, reperietur

$$= -\frac{(i+1)i}{2(2i+1)c}A$$

$$= +\frac{(i+2)(i+1)i(i-1)}{2\cdot 4(2i+1)^2c^2}A$$

$$= -\frac{(i+3)(i+2)(i+1)i(i-1)(i-2)}{2\cdot 4\cdot 6(2i+1)^3c^3}A$$

$$= +\frac{(i+4)(i+3)(i+2)(i+1)i(i-1)(i-2)(i-3)}{2\cdot 4\cdot 6\cdot 8(2i+1)^4c^4}A$$

etc.,

grale particulare erit:

$$\frac{\frac{i}{2i+1}Ax^{\frac{-i-1}{2i+1}} + \frac{i-1}{2i+1}Bx^{\frac{-i-2}{2i+1}} + \frac{i-2}{2i+1}Cx^{\frac{-i-3}{2i+1}} + \frac{i-3}{2i+1}Dx^{\frac{-i-4}{2i+1}} + \text{etc.}}{Ax^{\frac{i}{2i+1}} + Bx^{\frac{i-1}{2i+1}} + Cx^{\frac{i-2}{2i+1}} + Dx^{\frac{i-3}{2i+1}} + \text{etc.}}$$

undem denominatorem reducatur, statuamus:

$$\mathfrak{A} = cA$$

$$\mathfrak{B} = -\frac{i(i-1)}{2(2i+1)}A$$

$$\mathfrak{C} = +\frac{(i+1)i(i-1)(i-2)}{2\cdot 4(2i+1)^2c}A$$

$$\mathfrak{D} = -\frac{(i+2)(i+1)i(i-1)(i-2)(i-3)}{2\cdot 4\cdot 6(2i+1)^3c^2}A$$

etc.,

$$Ax^{2i+1} + Bx^{2i+1} + Cx^{2i+1} + Dx^{2i+1} + \text{etc.}$$

Ponamus porro brevitatis gratia:

$$Ax^{\frac{i}{2i+1}} + Bx^{\frac{i-1}{2i+1}} + Cx^{\frac{i-2}{2i+1}} + Dx^{\frac{i-3}{2i+1}} + \text{etc.} = P$$

$$Ax^{\frac{i}{2i+1}} - Bx^{\frac{i-1}{2i+1}} + Cx^{\frac{i-2}{2i+1}} - Dx^{\frac{i-3}{2i+1}} + \text{etc.} = Q$$

$$\mathfrak{A}x^{\frac{-i}{2i+1}} + \mathfrak{B}x^{\frac{-i-1}{2i+1}} + \mathfrak{C}x^{\frac{-i-2}{2i+1}} + \mathfrak{D}x^{\frac{-i-3}{2i+1}} + \text{etc.} =$$

$$-\mathfrak{A}x^{\frac{-i}{2i+1}} + \mathfrak{B}x^{\frac{-i-1}{2i+1}} - \mathfrak{C}x^{\frac{-i-2}{2i+1}} + \mathfrak{D}x^{\frac{-i-3}{2i+1}} - \text{etc.} =$$

atque integrale completum erit:

$$e^{-2(2i+1)cx^{\frac{i+1}{2i+1}}}\frac{Qy-\mathfrak{Q}}{Py-\mathfrak{P}} = \text{Const.}$$

Tum vero divisor, aequationem propositam reddens integ
$(Py-\mathfrak{P})(Qy-\mathfrak{Q})$.

COROLLARIUM 1

81. Quodsi ergo in aequatione

$$dy + yy\,dx + ax^{\frac{-4i}{2i+1}}dx = 0$$

coefficiens a fuerit quantitas negativa, ut posito $a = -cc$
realis, integrale completum hic inventum formam habet realem
facile exhiberi potest, pariter ac divisor, qui aequationem inte

COROLLARIUM 2

82. At si a fuerit quantitas positiva, puta $a = aa$, u
aequatio:

$$dy + yy\,dx + aax^{\frac{-4i}{2i+1}}dx = 0,$$

erit $c = a\sqrt{-1}$, et coefficientes B, D, F etc. et $\mathfrak{A}$, $\mathfrak{C}$, $\mathfrak{E}$ etc.
unde valores particulares $y = \frac{\mathfrak{P}}{P}$ et $y = \frac{\mathfrak{Q}}{Q}$ prodibunt imagin

COROLLARIUM 3

...e tamen casu, quo $c = a\sqrt{-1}$ et $cc = -aa$, fient $P+Q$ et ... quantitates reales, at $P-Q$ et $\mathfrak{P}-\mathfrak{Q}$ imaginariae. Quodsi ergo

$$= 2R,\quad P-Q = 2S\sqrt{-1},\quad \mathfrak{P}+\mathfrak{Q} = 2\mathfrak{R} \text{ et } \mathfrak{P}-\mathfrak{Q} = 2\mathfrak{S}\sqrt{-1}$$

..., $\mathfrak{R}$ et $\mathfrak{S}$ quantitates reales, et ob

$$= R+S\sqrt{-1},\quad Q = R-S\sqrt{-1},\quad \mathfrak{P} = \mathfrak{R}+\mathfrak{S}\sqrt{-1},\quad \mathfrak{Q} = \mathfrak{R}-\mathfrak{S}\sqrt{-1}$$

...r, reddetur aequationem integrabilem,

$$(RR+SS)yy - 2(R\mathfrak{R}+S\mathfrak{S})y + \mathfrak{R}\mathfrak{R}+\mathfrak{S}\mathfrak{S}$$

...realis.

COROLLARIUM 4

At eodem casu $c = a\sqrt{-1}$, ob

$$e^{p\sqrt{-1}} = \cos p + \sqrt{-1}\sin p,$$

$$\ldots = \cos 2(2i+1)ax^{\frac{1}{2i+1}} + \sqrt{-1}\sin 2(2i+1)ax^{\frac{1}{2i+1}};$$

...posito brevitatis gratia

$$2(2i+1)ax^{\frac{1}{2i+1}} = p,$$

...egrale completum:

$$(\cos p + \sqrt{-1}\sin p)\frac{(R+S\sqrt{-1})y - \mathfrak{R}+\mathfrak{S}\sqrt{-1}}{(R-S\sqrt{-1})y - \mathfrak{R}-\mathfrak{S}\sqrt{-1}} = \text{Const.},$$

...tamen est imaginaria.

COROLLARIUM 5

... Tribuatur autem constanti talis forma: $\alpha + \beta\sqrt{-1}$, et aequatione ... evoluta, erit:

$$(Ry-\mathfrak{R})\cos p - (Ry-\mathfrak{R})\sin p\sqrt{-1} + (Sy-\mathfrak{S})\cos p\sqrt{-1} - (Sy-\mathfrak{S})\sin p$$
$$= (Ry-\mathfrak{R})\alpha - (Ry-\mathfrak{R})\beta\sqrt{-1} + (Sy-\mathfrak{S})\alpha\sqrt{-1} + (Sy-\mathfrak{S})\beta.$$

...sequentur seorsim partes reales et imaginariae:

$$(Ry-\mathfrak{R})\cos p - (Sy-\mathfrak{S})\sin p = \alpha(Ry-\mathfrak{R}) + \beta(Sy-\mathfrak{S})$$
$$(Ry-\mathfrak{R})\sin p + (Sy-\mathfrak{S})\cos p = \beta(Ry-\mathfrak{R}) - \alpha(Sy-\mathfrak{S}),$$

Sit ergo $\alpha = \cos.\zeta$, et $\beta = \sin.\zeta$, prodibitque ex utraque

$$\frac{Ry - \mathfrak{R}}{Sy - \mathfrak{S}} = \frac{\sin.p + \sin.\zeta}{\cos.p + \cos.\zeta} = \cot.\frac{\zeta - p}{2}.$$

COROLLARIUM 6

86. Sumto ergo pro ζ angulo quocunque, si sit $c = a\sqrt{-1}$, completum aequationis propositae

$$\frac{Ry - \mathfrak{R}}{Sy - \mathfrak{S}} = \cot.\frac{\zeta - p}{2}$$

seu

$$y = \frac{\mathfrak{R}\sin.\frac{\zeta - p}{2} - \mathfrak{S}\cos.\frac{\zeta - p}{2}}{R\sin.\frac{\zeta - p}{2} - S\cos.\frac{\zeta - p}{2}}$$

existente $p = 2(2i+1)ax^{\frac{1}{2i+1}}$.

PROBLEMA 13

87. Denotante i numerum quemcunque integrum exhibere huius aequationis:

$$dy + yydx - ccx^{\frac{-4i}{2i-1}}dx = 0.$$

SOLUTIO

Quia est $n = \frac{i}{2i-1}$, haec resolutio derivari potest ex sol dentis problematis, ponendo $-i$ loco i. Quare tribuantur litteri sequentes valores:

$$B = +\frac{i(i-1)}{2(2i-1)c}A$$

$$C = +\frac{(i+1)i(i-1)(i-2)}{2\cdot 4(2i-1)^2c^2}A$$

$$D = +\frac{(i+2)(i+1)i(i-1)(i-2)(i-3)}{2\cdot 4\cdot 6(2i-1)^3c^3}$$

etc.

alterarum ...

$$\mathfrak{A} = cA$$
$$\mathfrak{B} = +\frac{(i+1)i}{2(2i-1)}A$$
$$\mathfrak{C} = +\frac{(i+2)(i+1)i(i-1)}{2\cdot 4(2i-1)^2 c}A$$
$$\mathfrak{D} = +\frac{(i+3)(i+2)(i+1)i(i-1)(i-2)}{2\cdot 4\cdot 6(2i-1)^3 c^2}A$$
etc.

valoribus constitutis, ponatur brevitatis gratia:

$$Ax^{\frac{+i}{2i-1}} + Bx^{\frac{+i+1}{2i-1}} + Cx^{\frac{+i+2}{2i-1}} + Dx^{\frac{+i+3}{2i-1}} + \text{etc.} = P$$
$$Ax^{\frac{+i}{2i-1}} - Bx^{\frac{+i+1}{2i-1}} + Cx^{\frac{+i+2}{2i-1}} - Dx^{\frac{+i+3}{2i-1}} + \text{etc.} = Q$$
$$\mathfrak{A}x^{\frac{-i}{2i-1}} + \mathfrak{B}x^{\frac{-i+1}{2i-1}} + \mathfrak{C}x^{\frac{-i+2}{2i-1}} + \mathfrak{D}x^{\frac{-i+3}{2i-1}} + \text{etc.} = \mathfrak{P}$$
$$-\mathfrak{A}x^{\frac{-i}{2i-1}} + \mathfrak{B}x^{\frac{-i+1}{2i-1}} - \mathfrak{C}x^{\frac{-i+2}{2i-1}} + \mathfrak{D}x^{\frac{-i+3}{2i-1}} - \text{etc.} = \mathfrak{Q}$$

...ine statim habentur duae integrationes particulares:

$$\text{I. } y = \frac{\mathfrak{P}}{P} \quad \text{et} \quad \text{II. } y = \frac{\mathfrak{Q}}{Q}.$$

...ero aequatio integralis completa erit:

$$e^{2(2i-1)cx^{\frac{-1}{2i-1}}}\frac{Qy-\mathfrak{Q}}{Py-\mathfrak{P}} = \text{Const.}$$

...isor aequationem propositam integrabilem reddens, fiet $=$
...$-\mathfrak{P})(Qy-\mathfrak{Q})$.

COROLLARIUM 1

...8. Quodsi autem aequatio proposita fuerit huiusmodi:

$$dy + yydx + aax^{\frac{-4i}{2i-1}}dx = 0,$$

... $cc = -aa$ et $c = a\sqrt{-1}$, integrationes particulares exhibitae fient ...inariae, ob B, D, F etc. item $\mathfrak{A}$, $\mathfrak{C}$, $\mathfrak{E}$ etc. imaginarias, dum reliquarum ...arum valores sunt reales.

COROLLARIUM 2

89. At si ponatur

$$P+Q=2R,\ P-Q=2S\sqrt{-1},\ \mathfrak{P}+\mathfrak{Q}=2\mathfrak{R} \text{ et } \mathfrak{P}-\mathfrak{Q}=2\mathfrak{S}$$

quantitates R, S, $\mathfrak{R}$ et $\mathfrak{S}$ nihilo minus fient, ut ante, reales, et divisor a
tionem reddens integrabilem erit:

$$(RR+SS)\,yy-2(R\mathfrak{R}+S\mathfrak{S})\,y+\mathfrak{R}\mathfrak{R}+\mathfrak{S}\mathfrak{S}.$$

COROLLARIUM 3

90. Tum vero, si ponatur brevitatis causa

$$2(2i-1)\,ax^{\frac{-1}{2i-1}}=p,$$

aequatio integralis completa erit:

$$\frac{Ry-\mathfrak{R}}{Sy-\mathfrak{S}}=\cot.\frac{\zeta+p}{2},$$

unde elicitur:

$$y=\frac{\mathfrak{R}\sin.\frac{\zeta+p}{2}-\mathfrak{S}\cos.\frac{\zeta+p}{2}}{R\sin.\frac{\zeta+p}{2}-S\cos.\frac{\zeta+p}{2}}$$

ubi angulus ζ vicem gerit constantis arbitrariae.

SCHOLION

91. Solutiones horum duorum postremorum problematum non ta
accuratam analysin sunt evolutae, quam per inductionem ex casibus pa
laribus supra expeditis derivatae, quandoquidem progressio ab his casil
sequentes satis erat manifesta. Fundamentum autem harum solutionum
potissimum est situm, quod solutio particularis, unde omnia sunt dedu
vera est geminata, cum quantitas c, cuius quadratum tantum in aequ
differentiali occurrit, aeque negative, ac positive, accipi possit. Quoties
huiusmodi aequationum binae solutiones particulares sunt cognitae,
multo facilius solutio generalis, indeque multiplicatores, eas integrabil
dentes, erui possunt, id quod operae pretium erit clarius exposuisse.

$$dy + Pydx + Qyydx + Rdx = 0$$

us solutionem generalem, et multiplicatorem, qui eam integrabilem

SOLUTIO

M et N huiusmodi functiones ipsius x, quae loco y substitutae, ambae propositae satisfaciant, ita ut sit:

$$dM + PMdx + QM^2dx + Rdx = 0$$

$$dN + PNdx + QN^2dx + Rdx = 0.$$

$$\frac{y-M}{y-N} = z \text{ seu } y = \frac{M-Nz}{1-z},$$

$$dy = \frac{dM - zdM + Mdz - Ndz - zdN + zzdN}{(1-z)^2},$$

loribus in aequatione proposita substitutis, et tota aequatione per ultiplicata, prodibit:

$$M - z(1-z)\,dN + (M-N)dz + P(1-z)\,Mdx - P(1-z)\,Nzdx$$
$$QMMdx - 2QMNzdx + QNNzzdx + R(1-z)^2dx = 0.$$

M et dN substituantur valores ex binis superioribus differentialibus

$$- z)\,Mdx - Q\,(1-z)\,M^2dx - R\,(1-z)\,dx$$
$$- z)\,Ndx + Qz(1-z)\,N^2dx + Rz(1-z)\,dx + (M-N)\,dz = 0$$
$$- z)\,Mdx + QM^2dx \qquad + R\,(1-z)^2dx$$
$$- z)\,Ndx - 2QMNzdx$$
$$+ QN^2zzdx,$$

atione in ordinem redacta, orietur:

$$QzM^2dx + QzN^2dx - 2QMNzdx + (M-N)\,dz = 0$$

$$Q(M-N)\,dx + \frac{dz}{z} = 0,$$

$$z = Ce^{-\int Q(M-N)dx},$$

unde aequatio integrata generalis erit:

$$e^{\int Q(M-N)dx} \frac{y-M}{y-N} = \text{Const.}$$

Pro multiplicatore autem inveniendo notetur, aequationem propositam substitutione primum per $(1-z)^2$ esse multiplicatam, tum vero divisam $z(M-N)$ evasisse integrabilem. Statim ergo per $\frac{(1-z)^2}{(M-N)z}$ multiplicat integrabilis: ex quo factor erit $\frac{(1-z)^2}{(M-N)z}$, qui ob $z = \frac{y-M}{y-N}$ hanc induet for

$$\frac{M-N}{(y-M)(y-N)}.$$

PROBLEMA 15

93. Proposita aequatione[1])

$$ydy + Pydx + Qdx = 0,$$

invenire conditiones functionum P et Q, ut huiusmodi multiplicator $(y$ - eam reddat integrabilem.

SOLUTIO

Ex natura ergo differentialium esse oportet:

$$\frac{1}{dx} d\cdot y(y+M)^n = \frac{1}{dy} d\cdot (Py+Q)\ (y+M)^n,$$

unde cum M sit functio ipsius x tantum, erit

$$ny(y+M)^{n-1}\frac{dM}{dx} = P(y+M)^n + n(Py+Q)\ (y+M)^{n-1},$$

quae divisa per $(y+M)^{n-1}$ abit in hanc:

$$\frac{nydM}{dx} = (n+1)Py + PM + nQ,$$

---

1) Cf. Commentationem 430 (indicis *Enestroemiani*). *Observationes circa aequationem tialem* $ydy + Mydx + Ndx = 0$. Novi Comment. acad. Petrop. 17, 1773, p. 105. Cf. quoqu *tiones calculi integralis*, vol. I, § 493—527. *Leonhardi Euleri Opera omnia*, series I, vol. 23 et 1

$$P = \frac{ndM}{(n+1)dx} \quad \text{et} \quad Q = \frac{-PM}{n} = -\frac{MdM}{(n+1)dx}.$$

loribus substitutis aequatio

$$ydy + \frac{nydM}{n+1} - \frac{MdM}{n+1} = 0$$

is, si multiplicetur per $(y+M)^n$.

COROLLARIUM 1

ia haec aequatio est homogenea, ea quoque fit integrabilis, si er

$$\text{ι} + 1)\,yy + nyM - MM = (y+M)((n+1)\,y - M).$$

hinc novae aequationes methodo hac tractabiles obtinentur.

COROLLARIUM 2

uoniam autem habemus duos multiplicatores

$$(y+M)^n \quad \text{et} \quad \frac{1}{(y+M)((n+1)\,y - M)},$$

r alterum dividatur, quoties constanti arbitrariae aequatus dabit ompletum. Quare aequatio

$$ydy + \frac{nydM}{n+1} - \frac{MdM}{n+1} = 0$$

r integrata praebet:

$$(y+M)^{n+1}((n+1)\,y - M) = \text{Const.}$$

PROBLEMA 16

Proposita aequatione

$$ydy + Pydx + Qdx = 0,$$

conditiones functionum P et Q, ut huiusmodi multiplicator

$$(yy + My + N)^n$$

dat integrabilem.

Ex natura differentialium sit necesse est:

$$\frac{1}{dx} d\cdot y(yy + My + N)^n = \frac{1}{dy} d\cdot (Py + Q)\ (yy +$$

Cum igitur M, N, P et Q sint per hypothesin functiones
evolutione:

$$ny\,(yy + My + N)^{n-1}\Big(y\frac{dM}{dx} + \frac{dN}{dx}\Big)$$

$$= P(yy + My + N)^n + n\,(Py + Q)\ (2y + M)\ (yy$$

et post divisionem per $(yy + My + N)^{n-1}$:

$$nyy\frac{dM}{dx} + \frac{nydN}{dx} = (2n + 1)\ Pyy + (n + 1)\ P$$
$$+ 2nQy$$

Hinc fieri oportet:

$$\text{I.}\quad ndM = (2n + 1)\ Pdx$$
$$\text{II.}\quad ndN = (n + 1)\ PMdx + 2nQ$$
$$\text{III.}\quad 0 = PN + nQM\,.$$

Prima dat

$$P = \frac{ndM}{(2n+1)\,dx}$$

et ultima

$$Q = \frac{-PN}{nM}\ \text{ seu }\ Q = \frac{-NdM}{(2n + 1)\ Mdx}\,,$$

qui valores in media substituti praebent:

$$ndN = \frac{n\,(n+1)\,MdM}{2n+1} - \frac{2nNdM}{(2n+1)}$$

seu

$$(2n + 1)\ MdN + 2NdM = (n + 1)\ M$$

quae multiplicata per $M^{\frac{-2n+1}{2n+1}}$ et integrata praebet:

$$(2n + 1)\ M^{\frac{2}{2n+1}}\,N = \text{Const.} + (n + 1)\int$$

seu

$$N = aM^{\frac{-2}{2n+1}} + \frac{1}{4}M^2.$$

$$Pdx = \frac{ndM}{2n+1} \quad \text{et} \quad Qdx = -\frac{aM^{\frac{-2n-3}{2n+1}}dM}{2n+1} - \frac{MdM}{4(2n+1)},$$

differentialis:

$$ydy + \frac{nydM}{2n+1} - \frac{MdM}{4(2n+1)} - \frac{a}{2n+1} M^{\frac{-2n-3}{2n+1}} dM = 0$$

dditur, si multiplicetur per

$$\left(yy + My + \frac{1}{4}M^2 + aM^{\frac{-2}{2n+1}}\right)^n.$$

COROLLARIUM 1

erit

$$\frac{-2n-3}{2n+1} = 1 \quad \text{seu} \quad n = -1,$$

rentialis est homogenea, et si

$$\frac{-2n-3}{2n+1} = 0 \quad \text{seu} \quad n = -\frac{3}{2},$$

Utroque autem casu nulla est difficultas, cum aequatio facile t.

COROLLARIUM 2

is ergo abstrusi erunt casus, quibus exponens $\frac{-2n-3}{2n+1}$ neque 1. Sit ergo

$$\frac{-2n-3}{2n+1} = m, \quad \text{unde fit} \quad 2n = \frac{-m-3}{m+1},$$

differentialis

$$(m+3)\, ydM + \frac{1}{8}(m+1)\, MdM + \frac{1}{2}a(m+1)\, M^m dM = 0$$

COROLLARIUM 3

99. Quod si iam pro M functiones quaecunque ipsius a
aequationes tam complicatae formari poterunt, quas quomod
tractari oporteat, vix liquet, cum tamen hac methodo earum
promtu.

SCHOLION

100. Si quis haec vestigia ulterius prosequi voluerit, dubi
quin haec methodus mox multo maiora sit acceptura incremen
versa Analysis non mediocriter promoveatur. Specimina eti
ita sunt comparata, ut viam ad investigationes profundiores pa
praecipue si insuper alia aequationum differentialium genera s
tractentur. Verum haec, quae hactenus protuli, sufficere v
Geometrarum ad ampliorem huius methodi enucleationem i
scopum mihi equidem potissimum proposueram.

CONSTRUCTIO AEQUATIONIS DIFFERENTIO-DIFFERENTIALIS $Ay\,du^2 + (B + Cu)\,du\,dy + (D + Eu + Fuu)\,ddy = 0$ SUMTO ELEMENTO du CONSTANTE

Commentatio 274 indicis ENESTROEMIANI
Novi Commentarii academiae scientiarum Petropolitanae 8 (1760/1), 1763, p. 150—166
Summarium ibidem p. 23—24

SUMMARIUM

Forma aequationis, quam Auctor hic construendam suscepit, ita est compara... cum in ...tissimo pateat, ac per universam Analysin amplissimum habeat usum; ... a, quae olim de celeberrima illa aequatione Riccatiana sunt investigata, contino... Si hoc negotium per methodos usitatas tentetur, summae difficultates obstant, ... s ad finem perduci queat; novam igitur Auctor ac prorsus singularem method... nit huiusmodi aequationes tractandi, cuius quidem iam pridem nonnulla egr... mina edidit; neque ullum est dubium, quin ista methodus, si diligentius excola... nia incrementa Analysi sit allatura. Casu autem evenit, ut haec tractatio non per... nem sit perducta, sive quaedam capita perierint, sive ab Auctore sint neglecta. ... m hic proferuntur, omnino sufficiunt ad vim novae huius methodi perspicien... e adeo, quae desunt, ab attento lectore harum rerum studioso haud difficulter ... tur. Quin etiam si ex hac parte attentio excitetur, nullum est dubium, quin Ana... multo maiora incrementa sit consecutura.

1. Aequationem hanc differentio-differentialem latissime patere, ex p... s formis[1]), in quas eam transmutare licet, facile intelligitur; plerun...

1) Vide Commentationem 678 voluminis I 23.

aequationem differentialem primi gradus:

$$dz + \frac{(B + Cu)zdu}{D + Eu + Fuu} + zzdu + \frac{Adu}{D + Eu + Fuu} = 0,$$

quae deinceps ad alias substitutiones amplissimum campum pa ob rem non parum Analysi consultum fore arbitror, si in gener tionis constructionem docuero, id quod per ea, quae olim RICCATIANA proposui, sequentem in modum praestari poterit[1]

2. Concipio autem y determinari formula quapiam inte quantitatem u novam variabilem x involvente, ita ut in hac in x ut variabilis, quantitas u vero ut constans tractetur. Cum aut sive analytice, sive per constructionem quadraturarum, fuerit a titati x valor quidam constans datus tribuitur, quo facto integ tabit functionem quandam ipsius u, quae sit ea ipsa, quam aequ exigit. Totum ergo negotium huc redit, ut formula illa integra u et x involvens inveniatur, quae hoc modo tractata verum va exhibeat.

3. Ponamus ergo esse

$$y = \int Pdx\,(u + x)^n,$$

in qua formula P denotet functionem quandam ipsius x ab u im quidem demum definiri oportet. Quae cum fuerit cognita, in per quadraturas concedetur, idque pro quocunque valore ipsi integratione ut constans spectatur. Tum integrali ita sumto, u valore ipsi x tributo evanescat, statuatur pro x alius quispiam et constans, ab u scilicet non pendens; quo facto aequabitur y fu determinatae ipsius u, quae sit ea ipsa, qua aequatio proposita

4. Etsi autem in integratione $\int Pdx\,(u + x)^n$ quantitas u habetur, tamen eius incrementum assignari potest, quod capit, s tur $u + du$, et integratio simili modo absolvatur. Ex princip

1) Vide Commentationes 31, 70 huius voluminis, p. 10 et p. 150.

2) Cf. Commentationes 44, 45, 70 huius voluminis, p. 36, 57, 150; vide quoque

ositis) cong...
formula eodem modo tractetur, ipsique x post integrationem valor determinatus tribuatur, cum fuerit

$$y = \int P dx\,(u + x)^n,$$

nunc, quatenus variato u simul y variationem subit,

$$dy = n du \int P dx\,(u + x)^{n-1}.$$

si porro simili modo differentiale ex variatione ipsius u ortum colligamus, du constans consequemur:

$$ddy = n(n-1) du^2 \int P dx (u + x)^{n-2}.$$

5. Cum igitur his integralibus modo praescripto ita sumtis, ut ipsi x valor uidam determinatus tribuatur, sicque ea in meras functiones ipsius u abeant, abeamus hos valores:

$$y = \int P dx (u + x)^n, \quad \frac{dy}{du} = n \int P dx (u + x)^{n-1}$$

t

$$\frac{ddy}{du^2} = n(n-1) \int P dx (u + x)^{n-2},$$

necesse est, ut vi aequationis propositae sit

$$\begin{aligned} A \int P dx\,(u + x)^n + n\,(B + Cu) \int P dx\,(u + x)^{n-1} \\ + n\,(n-1)\,(D + Eu + Fuu) \int P dx\,(u + x)^{n-2} = 0, \end{aligned}$$

in quibus integralibus sola x ut variabilis spectatur, u vero pro constant habetur. Haec autem aequatio tum solum locum habere debet, cum pos singulas integrationes quantitati x valor ille determinatus ab u non pender fuerit tributus.

6. In genere autem, antequam ipsi x iste valor assignatur, ista quantit non evanescet, sed potius cuipiam quantitati ex u et x compositae aequabitu quae autem ita comparata esse debet, ut illo casu, quo pro x valor ille dete minatus scribatur, evanescat. Sit igitur $R\,(u + x)^{n-1}$ ea quantitas indefini cui superior forma in genere aequetur, ubi R sit eiusmodi functio ipsius quae tam pro eo valore ipsius x, quo integralia singula evanescentia reddunt

H. I

1) Vide § 8 Commentationis 44 huius voluminis, p. 30.

causa, cur eos non statim determinaverim.

7. Quamdiu ergo x adhuc est variabilis, et u ut constans spectatur, est, ut expressio $R\,(u+x)^{n-1}$ aequetur huic formulae integrali:

$$\int P dx\,(u+x)^{n-2}\left(\begin{array}{lll} +Auu & +2Aux & +Axx \\ +nCuu & +nCux & +nBx \\ & +nBu & \\ +n(n-1)Fuu & +n(n-1)Eu & +n(n-1) \end{array}\right.$$

cuius propterea differentiale aequari oportet huic:

$$(u+x)^{n-2}\,(udR + xdR + (n-1)\,Rdx).$$

Quia autem R ab u pendere non debet, conditiones satisfacientes his ac nibus continentur:

$$A + nC + n\,(n-1)\,F = 0$$
$$dR = (2A + nC)\,Pxdx + n\,(B + (n-1)\,E)\,Pdx$$
$$xdR + (n-1)\,Rdx = APxxdx + nBPxdx + n\,(n-1)\,DPdx.$$

8. Si valor ipsius dR ex secunda in tertia substituatur, habebitu

$$(n-1)\,R = -(A + nC)\,Pxx - n\,(n-1)\,EPx + n\,(n-1)\,DI$$

et quia ex prima est

$$-A - nC = n\,(n-1)\,F,$$

prodit

$$R = nP\,(Fxx - Ex + D).$$

Deinde ob

$$2A + nC = -2n\,(n-1)\,F - nC$$

secunda induit hanc formam:

$$dR = nPdx\,(-(C + 2\,(n-1)\,F)\,x + B + (n-1)\,E),$$

quae per illam divisa dat:

$$\frac{dR}{R} = \frac{-(C + 2(n-1)F)xdx + (B + (n-1)E)dx}{Fxx - Ex + D};$$

$$Pdx = \frac{Rdx}{n(Fxx - Ex + D)},$$

n n per primam aequationem definitur, unde fit

$$n = \frac{F - C + \sqrt{((F-C)^2 - 4AF)}}{2F}.$$

ures casus perpendendi occurrunt, ac primo quidem ratione si is prodierit imaginarius, puta $n = \mu + \nu\sqrt{-1}$, notandum

$$r^{\sqrt{-1}} = \cos lr + \sqrt{-1}\ \sin lr,$$

$$r^n = r^\mu(\cos \nu lr + \sqrt{-1}\sin \nu lr),$$

rium exponentis ope sinuum ad imaginaria simplicia reducitur, ineeps eorum destructio mutua facilius perficietur. Deinde in-
uctionis R huc redigitur, ut sit

$$lR = -(n-1)l(Fxx - Ex + D) - \int\frac{Cxdx - Bdx}{Fxx - Ex + D},$$

ad hanc formam perducitur:

$$\left(n - 1 + \frac{C}{2F}\right)l(Fxx - Ex + D) + \left(B - \frac{CE}{2F}\right)\int\frac{dx}{Fxx - Ex + D}.$$

$B - \frac{CE}{2F} = 0$, videndum est, an formulae integrandae denomi-
$- Ex + D$ habeat duos factores simplices reales et inaequales, an
es; tum vero an in huiusmodi factores sit irresolubilis. Praeterea
$= 0$ peculiarem evolutionem postulat, quos diversos casus scorsim
.

I. CASUS QUO $B = \frac{CE}{2F}$.

equatio ergo resolvenda erit

$$Ay + \frac{C}{2F}(E + 2Fu)\frac{dy}{du} + (D + Eu + Fuu)\frac{ddy}{du^2} = 0,$$

i sumamus $y = \int Pdx(u+x)^n$, habemus primo

$$R = (D - Ex + Fxx)^{-n+1-\frac{C}{2F}},$$

neque

$$Pdx = \frac{1}{n} dx (D - Ex + Fxx)^{-n-\frac{C}{2F}},$$

a ut sit

$$y = \frac{1}{n} \int \frac{dx(u+x)^n}{(D - Ex + Fxx)^{+n+\frac{C}{2F}}}$$

uod integrale eiusmodi terminis ipsius x comprehendi debet, quibus qua

$$(u+x)^{n-1}(D - Ex + Fxx)^{-n+1-\frac{C}{2F}}$$

vanescat.

11. Quoties ergo formula $D - Ex + Fxx$ duos factores habet real uplici casu evanescit, unde bini integrationis termini constitui possu oc autem necesse est, ut eius exponens $-n+1-\frac{C}{2F}$, qui fit

$$= \frac{F \mp \sqrt{((F-C)^2 - 4AF)}}{2F},$$

it positivus, quia alioquin quantitas illa, cui formula proposita ae tatuitur, non in nihilum abiret. Hoc igitur casu constructio aequa ullam habebit difficultatem, propterea quod ob signum ambiguum expo omper valor positivus tribui potest. Sit enim exponens ille $= m$, et hab

$$4FFmm - 4FFm + 4AF + 2CF - CC = 0,$$

quae aequatio si habet radices reales, ob terminum $-4FFm$ negat altera certe erit positiva. Quem casum diligenter prosequamur.

12. Sit $D = aa$, $E = 0$ et $F = -1$, ita ut haec aequatio sit resolv

$$Ay + \frac{Cudy}{du} + (aa - uu)\frac{ddy}{du^2} = 0,$$

eritque

$$n = \frac{1 + C \pm \sqrt{(1 + 2C + CC + 4A)}}{2},$$

per est realis, nisi A sit quantitas negativa maior quam $\frac{1}{4}(1 + C)^2$:

$$m = -n + 1 + \frac{1}{2}C = \frac{1 \mp \sqrt{(1 + 2C + CC + 4A)}}{2},$$

positivo sumto, erit pro resolutione nostrae aequationis

$$y = \frac{1}{n}\int dx\,(u + x)^n\,(aa - xx)^{m-1},$$

le ita capiatur, ut posito $x = a$ evanescat; tum vero statuatur pro y prodibit functio ipsius u aequationi satisfaciens. Prout iam rus realis vel imaginarius, sequentia exempla subiungamus.

emplum 1. Sit $C = 2$ et $A = -2$, ut proposita sit haec aequatio:

$$-2y + \frac{2u\,dy}{du} + \frac{(aa - uu)\,ddy}{du^2} = 0,$$

et $m = 1$, unde fit

$$y = \int dx\,(u + x)$$

$$-2y + \frac{2u\,dy}{du} + \frac{(aa - uu)\,ddy}{du^2} = aa - xx$$

psius y ita absolvi debet, ut pro terminis integralis $aa - xx$ evanes- st si fuerit $x = a$ et $x = -a$. Fiet ergo

$$y = ux + \frac{1}{2}xx - au - \frac{1}{2}aa,$$

iam $x = -a$, erit $y = -2au$, qui valor aequationi utique satis- generalius quidem $y = au$, ex quo porro integrale completum eruitur, $y = uz$, unde fit

$$2aa\,du\,dz + (aa - uu)\,u\,ddz = 0, \text{ seu } \frac{ddz}{dz} + \frac{2aa\,du}{u(aa - uu)} = 0$$

$$\frac{uudz}{aa - uu} = \beta du,$$

rroque

$$z = \gamma - \beta u - \frac{\beta aa}{u}$$

nsequenter

$$y = \gamma u - \beta uu - \beta aa\text{[1]}.$$

---

1) Altera pars huius dissertationis periit. Confer praeter summarium litteras adhuc in
Eulero ad G. F. Muellerum datas

die 27. Julii 1762: ... Ferner die Piece so pag. 156 aufhört ist auch noch lang nicht zu
d es muß auch wohl ein Bogen von meinem Manuscript oder noch mehr weggekommen ode
t werden seyn...

et die 21. Septembris 1762: Abhandlung Nr. VI so unvollständig, mag nur so bleiben, wo
n vorhandenen das folgende einigermaßen ersehet; zum wenigsten jenes durch dieses versta
rden kan. Man kan auch diese Abhandlung als in zwey Teile geteilt ansehen, davon nur der
diesem Tom. eingerückt war; und ich kan wohl den andern von neuem aufsetzen, und zu
genden Band einschicken. A

DE RESOLUTIONE AEQUATIONIS
$dy + ayy\,dx = bx^m\,dx$

Commentatio 284 indicis ENESTROEMIANI
Novi Commentarii academiae scientiarum Petropolitanae 9 (1762/3, 1764) p. 154—169
Summarium ibidem p. 18—21

SUMMARIUM

Aequatio haec, iam dudum a Comite RICCATI Geometris proposita, tanto studio
summis ingeniis est pertractata, ut vix quicquam novi circa eius resolutionem proferri
osse videatur. Statim quidem infiniti valores pro exponente m assumendi sunt observati
uibus integrale exhibere liceat, qui valores hac serie progrediuntur: $0, -4, -\frac{4}{3}$
$-\frac{8}{3}, -\frac{8}{5}, -\frac{12}{5}, -\frac{12}{7}, -\frac{16}{7}, -\frac{16}{9}$ etc., ac methodus, qua hi casus sunt evoluti, it
erat comparata, ut ex cognito cuiusque casus integrali integrale sequentis definiretu
neque adeo casuum posteriorum integralia exhiberi possent, nisi iam omnes antecedent
fuerint expediti. In hac autem dissertatione id praestatur, ut unica operatione omniu
illorum casuum integralia simul eruantur, indeque statim vel centesimi casus integra
assignari possit. Methodus, qua hoc commodi est assecutus, omnino est singularis. Do
primo aequationem propositam, ope certae substitutionis, in aliam, quae adeo differentia
secundi gradus involvit, transformat, eamque deinceps per seriem infinitam integr
quae autem series ita est comparata, ut supra memoratis casibus alicubi abrumpa
expressionemque finitam suppeditet, unde integrale quaesitum facillime colligatur. Ver
tamen omnia haec integralia nonnisi sunt particularia, neque totam vim aequatio
differentialis propositae exhauriunt, deinde etiam, quoties quantitas b est negati
imaginariis ita inquinantur, ut omni plane usu destituantur. Utrique incommodo
Auctor ita medetur, ut primo methodum exponat, ex cognito huiusmodi aequatio
integrali quopiam particulari integrale completum eliciendi, quod si quantitas b fu
positiva, quantitates exponentiales implicat: deinde vero ostendit, quomodo istae qu
tates exponentiales, quae, existente b negativo, fiunt imaginariae, per tangentes arc
circularium realiter exprimi queant. Denique cum methodus illa, ex integrali partic

ingreditur, quam aeque negativo, ac positivo, accipere licet. Alia igitur methodo uti
ius ope ex cognitis duobus integralibus particularibus integrale completum, sine
va integratione, concludi queat. Quod cum ab eo, quod priori methodo erat erut
screpare nequeat, ex utriusque collatione integrationem priori implicatam efficere li
de postremo hanc integrationem maxime memorabilem deducit, quod sit

$$\int e^{\frac{2ac}{n}x^n}\frac{dx}{uu} = \frac{Ce^{\frac{2ac}{n}x^n}(z-u)}{Cu\left(2acx^{n-1}uz+\frac{udz}{dx}-\frac{zdu}{dx}\right)},$$

i quantitates z et u per x ita definiuntur, ut sit:

$$z = x^{\frac{-n+1}{2}} + \frac{(nn-1)}{8nac}x^{\frac{-3n+1}{2}} + \frac{(nn-1)(9nn-1)}{8n\cdot 16na^2c^2}x^{\frac{-5n+1}{2}} + \text{etc.}$$

$$u = x^{\frac{-n+1}{2}} - \frac{(nn-1)}{8nac}x^{\frac{-3n+1}{2}} + \frac{(nn-1)(9nn-1)}{8n\cdot 16na^2c^2}x^{\frac{-5n+1}{2}} - \text{etc.}$$

m igitur hae formae z et u adeo in infinitum excurrere queant, eo magis est mirand
od formulae $e^{\frac{2ac}{n}x^n}\frac{dx}{uu}$ integrale, idque per expressionem satis simplicem, exhiberi po
m vero etiam hoc consuetae integralium formae adversari videtur, quod quant
nstans arbitraria C, per integrationem ingressa, quae alioquin nude adiicitur, hic
rmae integrali sit implicata. Quod singulare phaenomenon si attentius perpendi
ex patebit, integrationem illam veritati consentaneam esse non posse, nisi denomina
rs

$$2acx^{n-1}uz + \frac{udz - zdu}{dx}$$

erit quantitas constans, puta A; tum enim istud integrale in formam naturalem abi

$$e^{\frac{2ac}{n}x^n}\frac{z}{Au} - \frac{1}{AC}.$$

um autem res ita se habeat, hoc modo explicari potest: Quoniam quantitates z et
ries exprimuntur, easque ipsas, quae initio ex evolutione aequationis differentiali
ndi gradus sunt eruta, vicissim patet, eas ita pendere ab x, ut sit:

$$ddz + 2acx^{n-1}dxdz + (n-1)acx^{n-2}zdx^2 = 0$$

$$ddu - 2acx^{n-1}dxdu - (n-1)acx^{n-2}udx^2 = 0.$$

anc prior aequatio per u, posterior vero per z, multiplicetur, ac productorum differ
bit

$$uddz - zddu + 2acx^{n-1}dx(udz + zdu) + 2(n-1)acx^{n-2}uzdx^2 = 0,$$

manifesto ...

$$udz - zdu + 2\,acx^{n-1}uzdx = Adx.$$

facto $ac = \infty$, fiat $u = z = x^{\frac{-n+1}{2}}$ et $uz = x^{-n+1}$, evidens est, statui ac, sicque integratio superior abit in hanc formam:

$$\int e^{\frac{2ac}{n}x^n}\frac{dx}{uu} = e^{\frac{2ac}{n}x^n}\frac{z}{2acu} - \text{Const.},$$

m principiis est conformis, sed etiam, facta differentiatione, ob

$$udz - zdu = 2\,acdx\,(1 - x^{n-1}ux)$$

gregio confirmatur. Hinc autem iam aequationis

$$dy + ayydx = accx^m dx,$$

$2n - 2$, et quantitatis z valore per superiorem seriem expresso, integrale otius ita exhiberi poterit, ut sit:

$$y = cx^{n-1} + \frac{dz}{azdx} + \frac{2\,Cce^{\frac{-2ac}{n}x^n}}{z(z - Ce^{\frac{-2ac}{n}x^n}u)}$$

$$y = cx^{n-1} + \frac{dz}{azdx} + \frac{2c}{z(De^{\frac{2ac}{n}x^n}z - u)}$$

] est illa constans arbitraria per integrationem iniecta ad integrale completum lum.

PROBLEMA 1

nvenire numeros loco exponentis indefiniti m substituendos, ut valor algebraice per x definiri queat.

SOLUTIO[1]

natur

$$y = cx^{n-1} + \frac{dz}{azdx},$$

to dx constante, erit

$$dy = (n-1)cx^{n-2}dx + \frac{ddz}{azdx} - \frac{dz^2}{azzdx}.$$

Cf. L. EULERI Commentationem 95 huius voluminis p. 162 et *Institutiones calculi integralis*, § 929—966. Petr. 1769 = LEONHARDI EULERI *Opera omnia*, I 12, p. 147—176. H. D.

facta substitutione transibit aequatio proposita in hanc:

$$\frac{ddz}{azdx} + (n-1)cx^{n-2}dx + accx^{2n-2}dx + \frac{2\,cx^{n-1}dz}{z} = bx^m d$$

Fiat $m = 2\,n-2$ et $b = acc$, habebiturque

$$ddz + (n-1)\,acx^{n-2}zdx^2 + 2\,acx^{n-1}\,dxdz = 0,$$

quae ergo resultat ex hac aequatione propositae aequivalente

$$dy + ayydx = accx^{2n-2}dx$$

facta substitutione

$$y = cx^{n-1} + \frac{dz}{azdx}\cdot$$

Fingatur iam haec aequatio:

$$z = Ax^{\frac{-n+1}{2}} + Bx^{\frac{-3n+1}{2}} + Cx^{\frac{-5n+1}{2}} + Dx^{\frac{-7n+1}{2}} + \text{etc.}$$

eritque differentiando:

$$\frac{dz}{dx} = -\frac{(n-1)}{2}Ax^{\frac{-n-1}{2}} - \frac{(3n-1)}{2}Bx^{\frac{-3n-1}{2}} - \frac{(5n-1)}{2}Cx^{\frac{-5n-1}{2}} -$$

$$\frac{ddz}{dx^2} = +\frac{(nn-1)}{4}Ax^{\frac{-n-3}{2}} + \frac{(9nn-1)}{4}Bx^{\frac{-3n-3}{2}} + \frac{(25\,nn-1)}{4}Cx^{\frac{-5n}{2}}$$

Cum vero ex superiori aequatione per dx^2 divisa sit:

$$\frac{ddz}{dx^2} + \frac{2\,acx^{n-1}dz}{dx} + (n-1)acx^{n-2}z = 0\,,$$

si series assumta substituatur, prodibit sequens aequatio:

$$0 = \left\{\begin{array}{l}
\qquad\qquad + \frac{(nn-1)}{4} \quad Ax^{\frac{-n-3}{2}} + \frac{(9\,nn-1)}{4} \quad Bx \\
\qquad\qquad + \frac{(25\,nn-1)}{4} \;\; Cx^{\frac{-5n-3}{2}} + \frac{(49\,nn-1)}{4} \;\; Dx \\
-(n-1)acAx^{\frac{n-3}{2}} - (3n-1)acBx^{\frac{-n-3}{2}} - (5n-1)acCx \\
\qquad\qquad -(7n-1)acDx^{\frac{-5n-3}{2}} - (9n-1)acEx \\
+(n-1)acAx^{\frac{n-3}{2}} + \;\; (n-1)acBx^{\frac{-n-3}{2}} + \;\; (n-1)acCx \\
\qquad\qquad + \;\; (n-1)acDx^{\frac{-5n-3}{2}} + \;\; (n-1)acEx
\end{array}\right.$$

$$\ldots 1) \cdot \frac{A}{4ac} = \frac{(nn-1)}{2} \cdot \frac{A}{4nac}$$

$$\ldots -1) \cdot \frac{B}{4ac} = \frac{(nn-1)}{2} \frac{(9nn-1)}{4} \cdot \frac{A}{4^2 n^2 a^2 c^2}$$

$$\ldots -1) \cdot \frac{C}{4ac} = \frac{(nn-1)}{2} \frac{(9nn-1)}{4} \frac{(25nn-1)}{6} \cdot \frac{A}{4^3 n^3 a^3 c^3}$$

$$\ldots -1) \cdot \frac{D}{4ac} = \frac{(nn-1)}{2} \frac{(9nn-1)}{4} \frac{(25nn-1)}{6} \frac{(49nn-1)}{8} \cdot \frac{A}{4^4 n^4 a^4 c^4}$$

etc.

...tur ergo z per x sequenti modo:

$$\ldots x^{\frac{-n+1}{2}} + \frac{(nn-1)}{8} \frac{A}{nac} x^{\frac{-3n+1}{2}} + \frac{(nn-1)}{8} \frac{(9nn-1)}{16} \frac{A}{n^2a^2c^2} x^{\frac{-5n+1}{2}}$$

$$+ \frac{(nn-1)}{8} \frac{(9nn-1)}{16} \frac{(25nn-1)}{24} \frac{A}{n^3a^3c^3} x^{\frac{-7n+1}{2}} + \text{etc.}$$

...ubstituto resultabit valor quaesitus: $y = cx^{n-1}$

$$\left.\frac{\ldots x^{\frac{-n-1}{2}} + \frac{(3n-1)}{2} \frac{(nn-1)}{8} \frac{A}{nac} x^{\frac{-3n-1}{2}} + \frac{(5n-1)}{2} \frac{(nn-1)}{8} \frac{(9nn-1)}{16} \frac{A}{n^2a^2c^2} x^{\frac{-5n-1}{2}} + \text{etc.}}{\ldots x^{\frac{-n+1}{2}} + \frac{(nn-1)}{8} \frac{A}{nac} x^{\frac{-3n+1}{2}} + \frac{(nn-1)}{8} \frac{(9nn-1)}{16} \frac{A}{n^2a^2c^2} x^{\frac{-5n+1}{2}} + \text{etc.}}\right\}$$

...tore ac denominatore per $Ax^{\frac{-n-1}{2}}$ diviso: $y = cx^{n-1}$

$$\frac{\ldots nn-1)}{8} \frac{x^{-n}}{nac} + \frac{(5n-1)}{2} \frac{(nn-1)}{8} \frac{(9nn-1)}{16} \frac{x^{-2n}}{n^2a^2c^2} + \frac{(7n-1)}{2} \frac{(nn-1)}{8} \frac{(9nn-1)}{16} \frac{(25nn-1)}{24} \frac{x^{-3n}}{n^3a^3c^3} \ldots}{\ldots) \frac{x^{-n}}{nac} + \frac{(nn-1)}{8} \frac{(9nn-1)}{16} \frac{x^{-2n}}{n^2a^2c^2} + \frac{(nn-1)}{8} \frac{(9nn-1)}{16} \frac{(25nn-1)}{24} \frac{x^{-3n}}{n^3a^3c^3} \ldots}$$

...xpressio generaliter in infinitum excurrens fit finita, si fuerit

$$(2i+1)^2 nn - 1 = 0,$$

...numerum quemcunque integrum, hoc est, si fuerit

les i fuerit numerus integer.

$$ayydx = accx^{\frac{-4i-2\pm 2}{2i+1}}dx$$

initis poterit exhiberi, seu valor ipsius y per x

. sit $m = 2n - 2 = \frac{-4i}{2i+1}$, erit huius aequa-

$$+\, ayydx = accx^{\frac{-4i}{2i+1}}dx$$

icis expressum:

$$ayx = acx^{\frac{1}{2i+1}}$$

$$\frac{i^2-1)(i^2-4)}{2\cdot 4(2i+1)^3}\frac{x^{\frac{-2}{2i+1}}}{a^2c^2} - \frac{i(i^2-1)(i^2-4)(i^2-9)}{2\cdot 4\cdot 6(2i+1)^4}\frac{x^{\frac{-3}{2i+1}}}{a^3c^3} + \text{etc.}$$
$$\overline{\frac{(i^2-1)(i+2)}{2\cdot 4(2i+1)^2}\frac{x^{\frac{-2}{2i+1}}}{a^2c^2} - \frac{i(i^2-1)(i^2-4)(i+3)}{2\cdot 4\cdot 6(2i+1)^3}\frac{x^{\frac{-3}{2i+1}}}{a^3c^3} + \text{etc.}}$$

nominatorem reductione erit:

$$\frac{\frac{i(i^2-1)(i-2)}{2\cdot 4(2i+1)^2}\frac{x^{\frac{-1}{2i+1}}}{ac} - \frac{i(i^2-1)(i^2-4)(i-3)}{2\cdot 4\cdot 6(2i+1)^3}\frac{x^{\frac{-2}{2i+1}}}{a^2c^2} + \text{etc.}}{\frac{i(i^2-1)(i+2)}{2\cdot 4(2i+1)^2}\frac{x^{\frac{-2}{2i+1}}}{a^2c^2} - \frac{i(i^2-1)(i^2-4)(i+3)}{2\cdot 4\cdot 6(2i+1)^3}\frac{x^{\frac{-3}{2i+1}}}{a^3c^3} + \text{etc.}}$$

t $m = \frac{-4i-4}{2i+1}$, erit huius aequationis

$$+\, ayydx = accx^{\frac{-4i-4}{2i+1}}dx$$

expressum:

$$ayx = acx^{\overline{2i+1}}$$

$$\frac{i(i+1)(i+2)\,x^{\frac{1}{2i+1}}}{2(2i+1)^3\;ac} + \frac{i(i^2-1)(i+2)(i+3)\,x^{\frac{2}{2i+1}}}{2\cdot4(2i+1)^3\;a^2c^2} + \frac{i(i^2-1)(i^2-4)(i+3)(i+4)\,x^{\frac{3}{2i+1}}}{2\cdot4\cdot6(2i+1)^4\;a^3c^3} + \text{etc.}$$
$$\frac{i(i+1)\;x^{\frac{1}{2i+1}}}{2(2i+1)\;ac} + \frac{i(i^2-1)(i+2)\,x^{\frac{2}{2i+1}}}{2\cdot4(2i+1)^2\;a^2c^2} + \frac{i(i^2-1)(i^2-4)(i+3)\,x^{\frac{3}{2i+1}}}{2\cdot4\cdot6(2i+1)^3\;a^3c^3} + \text{etc.}$$

ad communem denominatorem reductione, erit $ayx =$

$$\frac{\dfrac{i+1)(i+2)}{2(2i+1)} + \dfrac{i(i+1)(i+2)(i+3)\,x^{\frac{1}{2i+1}}}{2\cdot4(2i+1)^2\;ac} + \dfrac{i(i^2-1)(i+2)(i+3)(i+4)\,x^{\frac{2}{2i+1}}}{2\cdot4\cdot6(2i+1)^3\;a^2c^2} + \text{etc.}}{\dfrac{i+1)\;x^{\frac{1}{2i+1}}}{i+1)\;ac} + \dfrac{i(i^2-1)(i+2)\,x^{\frac{2}{2i+1}}}{2\cdot4(2i+1)^3\;a^2c^3} + \dfrac{i(i^2-1)(i^2-4)(i+3)\,x^{\frac{3}{2i+1}}}{2\cdot4\cdot6(2i+1)^3\;a^3c^3} + \text{etc.}}$$

unque igitur fuerit i numerus integer, toties huius aequationis:

$$dy + ayydx = accx^{\frac{-4i-2+2}{2i+1}}dx$$

in terminis algebraicis potest exprimi. Q. E. I.

COROLLARIUM 1

Aequatio ergo proposita

$$dy + ayydx = accx^m dx$$

nem algebraicam admittit, si fuerit exponens m vel terminus huius

$$0,\; -\frac{4}{3},\; -\frac{8}{5},\; -\frac{12}{7},\; -\frac{16}{9},\; -\frac{20}{11},\; -\frac{24}{13},\; \text{etc.}$$

rit m terminus ex hac fractionum serie:

$$-\frac{4}{1},\; -\frac{8}{3},\; -\frac{12}{5},\; -\frac{16}{7},\; -\frac{20}{9},\; -\frac{24}{11},\; -\frac{28}{13},\; \text{etc.}$$

COROLLARIUM 2

ubstituamus in priori integrabilitatis classe loco i successive numeros , 4 etc. atque reperietur, ut sequitur.

ntegrale erit:

$$ayx = acx \text{ sivo } y = c.$$

Si $i = 1$, huius aequationis:

$$\text{II. } dy + ayydx = accx^{-\frac{4}{3}}dx$$

ntegrale erit:

$$ayx = \frac{acx^{\frac{1}{3}}}{1-\frac{1\cdot2}{2\cdot3}\cdot\frac{x^{-\frac{1}{3}}}{ac}} \text{ seu } y = \frac{cx^{-\frac{2}{3}}}{1-\frac{1}{3}\frac{x^{-\frac{1}{3}}}{ac}} = \frac{3\,acc}{3\,acx^{\frac{2}{3}}-x^{\frac{1}{3}}}.$$

Si $i = 2$, huius aequationis:

$$\text{III. } dy + ayydx = accx^{-\frac{8}{5}}dx$$

ntegrale erit:

$$ayx = \frac{acx^{\frac{1}{5}}-\frac{2\cdot1}{2\cdot5}}{1-\frac{2\cdot3}{2\cdot5}\cdot\frac{x^{-\frac{1}{5}}}{ac}+\frac{2\cdot3\cdot4}{2\cdot4\cdot5^2}\cdot\frac{x^{-\frac{2}{5}}}{a^2c^2}} = \frac{acx^{\frac{1}{5}}-\frac{1}{5}}{1-\frac{3\,x^{-\frac{1}{5}}}{5\,ac}+\frac{3\,x^{-\frac{2}{5}}}{5^2a^2c^2}}.$$

Si $i = 3$, huius aequationis:

$$\text{IV. } dy + ayydx = accx^{-\frac{12}{7}}\,dx$$

ntegrale erit:

$$ayx = \frac{acx^{\frac{1}{7}}-\frac{3\cdot2}{2\cdot7}+\frac{3\cdot2\cdot1\cdot4}{2\cdot4\cdot7^2}\cdot\frac{x^{-\frac{1}{7}}}{ac}}{1-\frac{3\cdot4}{2\cdot7}\cdot\frac{x^{-\frac{1}{7}}}{ac}+\frac{3\cdot4\cdot5\cdot2}{2\cdot4\cdot7^2}\cdot\frac{x^{-\frac{2}{7}}}{a^2c^2}-\frac{3\cdot4\cdot5\cdot6\cdot2}{2\cdot4\cdot6\cdot7^3}\cdot\frac{x^{-\frac{3}{7}}}{a^3c^3}}$$

sive

$$ayx = \frac{acx^{\frac{1}{7}}-\frac{3}{7}+\frac{3\cdot1}{7^2}\cdot\frac{x^{-\frac{1}{7}}}{ac}}{1-\frac{6}{7}\cdot\frac{x^{-\frac{1}{7}}}{ac}+\frac{3\cdot5}{7^2}\cdot\frac{x^{-\frac{2}{7}}}{a^2c^2}-\frac{1\cdot3\cdot5}{7^3}\cdot\frac{x^{-\frac{3}{7}}}{a^3c^3}}.$$

integrale erit:

$$ayx = \frac{acx^{\frac{1}{9}} - \frac{4\cdot3}{2\cdot9} + \frac{4\cdot3\cdot2\cdot5}{2\cdot4\cdot9^2}\cdot\frac{x^{-\frac{1}{9}}}{ac} - \frac{4\cdot3\cdot2\cdot1\cdot5\cdot6}{2\cdot4\cdot6\cdot9^3}\cdot\frac{x^{-\frac{2}{9}}}{a^2c^2}}{1 - \frac{4\cdot5}{2\cdot9}\cdot\frac{x^{-\frac{1}{9}}}{ac} + \frac{4\cdot5\cdot6\cdot3}{2\cdot4\cdot9^2}\cdot\frac{x^{-\frac{2}{9}}}{a^2c^2} - \frac{4\cdot5\cdot6\cdot7\cdot3\cdot2}{2\cdot4\cdot6\cdot9^3}\cdot\frac{x^{-\frac{3}{9}}}{a^3c^3} + \frac{4\cdot5\cdot6\cdot7\cdot8\cdot3\cdot2\cdot1}{2\cdot4\cdot6\cdot8\cdot9^4}\cdot\frac{x}{a^4c}}$$

Si $i = 5$, huius aequationis

$$\text{VI.}\quad dy + ayydx = accx^{-\frac{20}{11}}dx$$

integrale erit:

$$= \frac{acx^{\frac{1}{11}} - \frac{5\cdot4}{2\cdot11} + \frac{5\cdot4\cdot3\cdot6}{2\cdot4\cdot11^2}\frac{x^{-\frac{1}{11}}}{ac} - \frac{5\cdot4\cdot3\cdot2\cdot6\cdot7}{2\cdot4\cdot6\cdot11^3}\frac{x^{-\frac{2}{11}}}{a^2c^2} + \frac{5\cdot4\cdot3\cdot2\cdot1\cdot6\cdot7\cdot8}{2\cdot4\cdot6\cdot8\cdot11^4}\frac{x^{-\frac{3}{11}}}{a^3c^3}}{1 - \frac{5\cdot6}{2\cdot11}\frac{x^{-\frac{1}{11}}}{ac} + \frac{5\cdot6\cdot7\cdot4}{2\cdot4\cdot11^2}\frac{x^{-\frac{2}{11}}}{a^2c^2} - \frac{5\cdot6\cdot7\cdot8\cdot4\cdot3}{2\cdot4\cdot6\cdot11^3}\frac{x^{-\frac{3}{11}}}{a^3c^3} + \frac{5\cdot6\cdot7\cdot8\cdot9\cdot4\cdot3\cdot2}{2\cdot4\cdot6\cdot8\cdot11^4}\frac{x^{-\frac{4}{11}}}{a^4c^4} - \frac{5\cdot6\cdot7\cdot8\cdot9\cdot10}{2\cdot4\cdot6\cdot8\cdot10}}$$

COROLLARIUM 3

4. In posteriori integrabilitatis ordine substituamus pariter loco i num
0, 1, 2, 3, 4 etc. ac reperietur, ut sequitur.

Si $i = 0$, huius aequationis:

$$\text{I.}\quad dy + ayydx = accx^{-4}dx$$

integrale erit:

$$ayx = \frac{acx^{-1} + \frac{1\cdot2}{2\cdot1}}{1} = 1 + \frac{ac}{x} \text{ seu } y = \frac{1}{ax} + \frac{c}{xx}.$$

Si $i = 1$, huius aequationis:

$$\text{II.}\quad dy + ayydx = accx^{-\frac{8}{3}}dx$$

integrale erit:

$$ayx = \frac{acx^{-\frac{1}{3}} + \frac{2\cdot3}{2\cdot3} + \frac{1\cdot2\cdot3\cdot4}{2\cdot4\cdot3^2}\frac{x^{\frac{1}{3}}}{ac}}{1 + \frac{1\cdot2}{2\cdot3}\cdot\frac{x^{\frac{1}{3}}}{ac}} = \frac{acx^{-\frac{1}{3}} + 1 + \frac{x^{\frac{1}{3}}}{3ac}}{1 + \frac{x^{\frac{1}{3}}}{3ac}}.$$

$$ayx = \frac{acx^{\frac{1}{5}} + [\text{illegible}]}{1 + [\text{illegible}]}$$

$i = 3$, huius aequationis:

$$\text{IV. } dy + ayydx = acx^{-\frac{12}{7}}dx$$

integrale erit:

$$ayx = \frac{acx^{\frac{1}{7}} + \frac{4 \cdot 6}{2 \cdot 7} + [\text{illegible}]}{1 + \frac{3 \cdot 4}{2 \cdot 7} + [\text{illegible}]}$$

Ex his casibus analogia patet, cuius ope omnium casuum, qui quidem integrationem admittunt, integralia algebraica expedite formari possunt.

SCHOLION

De his integralibus autem probe notandum est, ea non esse completa neque adeo aeque late patere, ac aequationem differentialem, id quod vel ex casu

$$dy + ayydx = acdx$$

cui etsi satisfacit $y = c$, tamen facile intelligitur, logarithmos insuper comprehendi. Manifestum autem hoc est quoque hinc, quod in his integralibus non contineatur nova constans arbitraria, quae in differentiali non erat, in qua criterium integrationis completae versatur. Caeterum vero hinc tamen integralia cuiusvis casus obtinentur, eo quod c tam affirmative, quam negative, assumi licet, aequatione differentiali, quae tantum cc continet, non mutata.

PROBLEMA 2

Invento ope praecedentis methodi integrali particulari pro casibus aequationis $dy + ayydx = acx^m dx$, invenire integrale completum pro iisdem casibus[1]).

[1]) Vide notam p. 405. H. D.

Posito $m = 2n - 2$, integrale particulare aequationis propositae inven-
est esse $ayx = acx^n$

$$\frac{n-1)(nn-1)x^{-n}}{2 \cdot 8n \cdot ac} + \frac{(5n-1)(nn-1)(9nn-1)x^{-2n}}{2 \cdot 8n \cdot 16n \cdot a^2c^2} + \frac{(7n-1)(nn-1)(9nn-1)(25nn-1)x^{-3n}}{2 \cdot 8n \cdot 16n \cdot 24n \cdot a^3c^3} + \ldots}{1 + \frac{(nn-1)x^{-n}}{8n \cdot ac} + \frac{(nn-1)(9nn-1)x^{-2n}}{8n \cdot 16n \cdot a^3c^3} + \frac{(nn-1)(9nn-1)(25nn-1)x^{-3n}}{8n \cdot 16n \cdot 24n \cdot a^3c^3} + \ldots}$$

loco scribamus brevitatis gratia $y = P$. Cum igitur P sit eiusmodi valor,
ariabilem x datus, qui satisfaciat aequationi

$$dy + ayydx = accx^{2n-2}dx,$$

utique

$$dP + aP^2dx = accx^{2n-2}dx.$$

mus iam integrale completum aequationis propositae

$$dy + ayydx = accx^{2n-2}dx$$

$= P + v$, quo valore loco y substituto habebimus hanc aequationem

$$dP + dv + aP^2dx + 2aPvdx + avvdx = accx^{2n-2}dx.$$

vero sit

$$dP + aP^2dx = accx^{2n-2}dx,$$

$$dv + 2aPvdx + avvdx = 0.$$

$= \frac{1}{u}$, erit

$$du - 2aPudx = adx,$$

multiplicata per $e^{-2a\int Pdx}$ denotante e numerum, cuius logarithmus hyper-
s est $= 1$, fit integrabilis; erit scilicet aequationis

$$e^{-2a\int Pdx}(du - 2aPudx) = e^{-2a\int Pdx}adx$$

ale

$$e^{-2a\int Pdx}u = \int e^{-2a\int Pdx}adx;$$

o

$$u = e^{2a\int Pdx}\int e^{-2a\int Pdx}adx.$$

alore, cum sit $v = \frac{1}{u}$, substituto, erit integrale completum aequationis
itae

[illegible]

$$P = cx^{n-1} + \frac{dz}{azdx};$$

$$\frac{(nn-1)x^{[illegible]}}{8n\,ac} + \frac{(nn-1)(9nn-1)x^{[illegible]}}{8n \cdot 16n\,a^2c^2} + \text{[illegible]} + \text{etc.}$$

$$\int Pdx = \frac{cx^n}{n} + \frac{1}{a}lz \quad \text{et} \quad e^{[illegible]} = [illegible]$$

quo substituto habebitur integrale completum:

$$y = cx^{n-1} + \frac{dz}{azdx} + \frac{e^{\frac{2acx^n}{n}}}{zz\int e^{\frac{2acx^n}{n}}\,adx : zz}. \qquad \text{Q. E. I.}$$

ALITER

Quemadmodum hac ratione ex uno integrali particulari inventum integrale completum, ita ex duobus integralibus particularibus expeditius integrale completum indagabitur, neque in hoc modo pervenitur ad formulam integralem cuiusmodi est $\int e^{\frac{2acx^n}{n}}\,adx : zz$, quae integrali completo, quod in[illegible] involvitur. Cum enim aequatio

$$dy + ayydx = accx^{2n-2}dx$$

invariata, sive c affirmative, sive negative accipiatur, habemus utique integralia particularia, quorum prius est

$$y = P = cx^{n-1} + \frac{dz}{azdx},$$

$$[illegible]\,x^{\frac{[illegible]}{2}} + \frac{(nn-1)\,x^{[illegible]}}{8n\,ac} + \frac{(nn-1)(9nn-1)\,x^{[illegible]}}{8n \cdot 16n\,a^2c^2} + \text{etc.}$$

que

$$u = x^{\frac{n+1}{2}} - \frac{(nn-1)}{8n}\cdot\frac{x^{\frac{-3n+1}{2}}}{ac} + \frac{(nn-1)}{8n}\cdot\frac{(9nn-1)}{16n}\cdot\frac{x^{\frac{-5n+1}{2}}}{a^2c^2} - \text{etc.},$$

duo valores z et u tantum signis inter se differunt. Erit ergo tam

$$dP + aP^2dx = accx^{2n-2}dx,$$

m

$$dQ + aQ^2dx = accx^{2n-2}dx.$$

amus iam[1])

$$R = \frac{P-y}{Q-y},$$

e aequatio sit integralis completa propositae differentialis; quam form
assumimus, quia in ea utraque particularium $y = P$ et $y = Q$ contine
quemque si fiat $R = 0$, hoc si $R = \infty$. Fiet ergo $QR - Ry = P - y$ hinc

$$y = \frac{QR-P}{R-1},$$

e dat

$$dy = \frac{RRdQ - QdR - RdQ - RdP + dP + PdR}{(R-1)^2},$$

stituantur hic valores supra inventi

$$dP = -aP^2dx + accx^{2n-2}dx$$

$$dQ = -aQ^2dx + accx^{2n-2}dx,$$

que

$$accx^{2n-2}dx + \frac{aP^2dx}{R-1} - \frac{aQ^2Rdx}{R-1} + \frac{(P-Q)dR}{(R-1)^2} = -a\frac{(QR-P)^2dx}{(R-1)^2} + accx^{2n}$$

hac aequatione resultat haec

$$(P-Q)\,dR = aRdx\,(P-Q)^2,$$

e divisa per $R(P-Q)$ dat

1) Cf. L. Euleri Commentationem 269; vide p. 380 huius voluminis. II.

...nis existit, critque integrale

$$\sim lC = -\frac{2acx^n}{n} + lu - lz.$$

...rit

$$\frac{(\ldots x^{n-1}zdx + dz - ayzdx) : z}{(\ldots x^{n-1}udx + du - ayudx) : u} = \frac{Ce^{\frac{-2acx^n}{n}}u}{z}.$$

...um u et z per x constant, habebitur aequatio inte-

$$\frac{dz + acx^{n-1}zdx - ayzdx}{du - acx^{n-1}udx - ayudx} = \frac{(P-y)z}{(Q-y)u}. \qquad \text{Q. E. I.}$$

COROLLARIUM 1

...quem supra pro y invenimus, ita erat comparatus,

$$y = cx^{n-1} - \frac{(K+L)}{ax(M+N)};$$

$$\ldots\frac{\ldots-1)}{\ldots 3n}\cdot\frac{x^{-2n}}{a^2c^2} + \frac{(9n-1)}{2}\frac{(n^2-1)}{8n}\frac{(9n^2-1)}{16n}\frac{(25n^2-1)}{24n}\frac{(49n^2-1)}{32n}\cdot\frac{x^{-4n}}{a^4c^4} + \text{etc.}$$

$$\frac{(7n-1)}{2}\frac{(nn-1)}{8n}\frac{(9nn-1)}{16n}\frac{(25nn-1)}{24n}\cdot\frac{x^{-3n}}{a^3c^3} + \text{etc.}$$

$$\frac{(nn-1)}{n}\frac{(9nn-1)}{16n}\frac{(25nn-1)}{24n}\frac{(49nn-1)}{32n}\cdot\frac{x^{-4n}}{a^4c^4} + \text{etc.}$$

$$\frac{\ldots 5nn-1)}{24n}\cdot\frac{x^{-3n}}{a^3c^3} + \text{etc.}$$

...erit alter valor particularis

$$y = -cx^{n-1} - \frac{(K-L)}{ax(M-N)}.$$

$$dy + ayydx = acx^{2n-2}dx$$

...pletum fore:

$$C_e = \frac{(acx^n - axy)(M + N) - K - L}{(acx^n + axy)(M - N) - K + L}$$

...ato loco C

$$C_e = \frac{ax(cx^{n-1} - y)(M + N) - K - L}{ax(cx^{n-1} + y)(M - N) + K - L}$$

COROLLARIUM 2

... numerus negativus, fiet c hincque L et N quantitates imaginariae, ... $L\sqrt{-1}$ et $N\sqrt{-1}$ quantitates reales. Tum autem integrale ... aliter expressum erit:

$$\ldots = A\ \text{tang.}\ \frac{acx^n N - axyM - K}{acx^n M\sqrt{-1} - axyN\sqrt{-1} - L\sqrt{-1}}$$

COROLLARIUM 3

... $b\sqrt{-1}$, ut habeatur haec aequatio integranda:

$$dy + ayydx + abbx^{2n-2}dx = 0.$$

... aequationis integrale completum erit[1]):

$$C = \ldots A\ \text{tang.}\ \frac{abx^n N - axyM - K}{abx^n M - axyN - L}$$

$$C = \ldots A\ \text{tang.}\ \frac{K - abx^n N + axyM}{L - abx^n M + axyN};$$

H. D.

... analyticae

$$\frac{n-1)}{8n}\,\frac{(9nn-1)}{16n}\,\frac{(25nn-1)}{24n}\cdot\frac{x^{-3n}}{a^3b^3} + \text{etc.}$$

$$\frac{(nn-1)}{8n}\,\frac{(9nn-1)}{16n}\,\frac{(25nn-1)}{24n}\,\frac{(49nn-1)}{32n}\cdot\frac{x^{-4n}}{a^4b^4} - \text{etc.}$$

$$\frac{(25nn-1)}{24n}\cdot\frac{x^{-3n}}{a^3b^3} + \text{etc.}$$

particularia, quae simul sint algebraica, non

OROLLARIUM 4

$= \frac{\mp 1}{2i+1}$, denotante i numerum quemcunque
ılgebraicae pro litteris K, L, M et N reperiuntur.
ıequationis huius

$$- ayydx = accx^{2n-2}dx$$

ro aequationis

$$yydx + abbx^{2n-2}dx = 0$$

lvitur.

SCHOLION

differentialis propositae $dy + ayydx = accx^{2n-2}dx$
nodo expressimus, poterimus formulae integralis

$$\int \frac{e^{\frac{-2acx^n}{n}}dx}{zz},$$

ı ex posteriori assignare, huiusque adeo integra-
ıaximopere difficilis videatur, exhibere. Posteriori

$R = \frac{Ce^{\frac{2acx^n}{n}}u}{z}$, $P = cx^{n-1} + \frac{dz}{azdx}$ et $Q = cx^{n-1} + \frac{du}{audx}$.

...tenter habebitur

$$y = cx^{n-1} + \frac{dz}{azdx} + \frac{\left(2cx^{n-1} + \frac{dz}{azdx} - \frac{du}{audx}\right)Ce^{\frac{2acx^n}{n}}u}{z - Ce^{\frac{2acx^n}{n}}u}.$$

...orem vero integrationem est

$$y = cx^{n-1} + \frac{dz}{azdx} + \frac{e^{\frac{2acx^n}{n}}}{zz\int e^{\frac{2acx^n}{n}}adx:zz},$$

...um comparatione oritur

$$\frac{z - Ce^{\frac{2acx^n}{n}}u}{Czzu\left(2cx^{n-1} + \frac{dz}{azdx} - \frac{du}{audx}\right)} = \int\frac{e^{\frac{2acx^n}{n}}adx}{zz},$$

...mmutatur in hanc aequationem:

$$\frac{zdx - Ce^{\frac{2acx^n}{n}}udx}{Cz(2acx^{n-1}uzdx + udz - zdu)} = \int\frac{e^{\frac{2acx^n}{n}}dx}{zz}.$$

...ergo fuerit:

$$z = x^{\frac{n+1}{2}} + \frac{(nn-1)}{8n}\cdot\frac{x^{\frac{3n+1}{2}}}{ac} + \frac{(nn-1)(9nn-1)}{8n\cdot16n}\cdot\frac{x^{\frac{5n+1}{2}}}{a^2c^2} + \text{etc.}$$

$$u = x^{\frac{n+1}{2}} - \frac{(nn-1)}{8n}\cdot\frac{x^{\frac{3n+1}{2}}}{ac} + \frac{(nn-1)(9nn-1)}{8n\cdot16n}\cdot\frac{x^{\frac{5n+1}{2}}}{a^2c^2} - \text{etc.}$$

...rmula differentialis

integrari poterit eritque integrale[1])

$$dx \quad Ce^{\ldots} \quad adx$$
$$C{:}(2acx^{n-1}a\,dx + adl \quad da)$$ [illegible]

Simili vero modo facto c negativo, quo x et a inter se permutantur, erit t[illegible] differentialis

$$x^{\ldots}\,dx \over aa$$ [illegible]

integrale

$$adx \quad Ce^{\ldots}\,dx \qquad\qquad Ce^{\ldots}\,dx \quad adx$$
$$Ca(\;2acx^{n-1}a\,dx + dx\,a \quad adl\;) \quad Ca(2ax^{n-1}a\,dx \quad adl \quad d$$ [illegible]

in quibus integrationibus C denotat eam constantem arbitrariam, q[uae per] integrationem more solito ingreditur.

1) Vide p. 401.